Chiral polytopes whose smallest regular cover is a polytope

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Abstract

We give a criterion for when the smallest regular cover of a chiral polytope $\mathcal{P}$ is itself a polytope, using only information about the facets and vertex-figures of $\mathcal{P}$.

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1 Introduction

Abstract polytopes are combinatorial objects that generalize convex polytopes. Every finite abstract 3-polytope can be identified with a face-to-face tiling of a closed surface, though some such tilings do not yield abstract 3-polytopes. More generally, an abstract n-polytope can be thought of as a collection of abstract $(n-1)$-polytopes glued together in a 'nice' way that ensures that the resulting structure still resembles the incidence relation of a convex polytope.

The study of symmetry is central in the theory of abstract polytopes. The most symmetric polytopes are called *regular*, and there is a well-developed theory of regular polytopes; see [9] for an extensive overview. Polytopes that have full symmetry under abstract rotations but no symmetry by an abstract reflection are called *chiral*. Chiral polytopes have also been the subject of much study, but they have proven much more difficult to pin down than regular polytopes. Chiral polytopes were introduced in [14], building on earlier work of Coxeter and others on chiral maps and honeycombs. Since then, many papers have investigated the structure of chiral polytopes; see [1, 2, 5, 13] for a broad selection.

Every chiral polytope has a unique smallest regular polytope-like object that covers it. Our goal is to determine when the smallest regular cover of a chiral polytope is itself a polytope. (This is essentially Problem 3 in [13].) A partial answer is given in [11, Cor. 7.5], covering the case where the facets or vertex-figures are regular. Here we will give an answer for the case where the facets and vertex-figures are both chiral. Furthermore, our characterization only uses information about the facets and vertex-figures; we do not need to know any of the global structure of the chiral polytope in question.

2 Background

2.1 Polytopes

The basic notions of abstract polytopes can be found in [9] and chiral polytopes are described in [14]. Here we review the necessary background.

Suppose $\mathcal{P}$ is a partially-ordered set with a unique minimal element F_{-1} and a unique maximal element F_n. If the maximal chains of $\mathcal{P}$ all have the same length, then $\mathcal{P}$ is a *graded poset*, where each element has a rank equal to 1 more than the maximum rank of all elements it covers. In our treatment, we give the minimal element a rank of -1. If $\mathcal{P}$ also satisfies the two conditions below, it is an *(abstract) n-polytope*:

(a) (Diamond condition): Whenever $F < G$ and $\text{rank}(G) - \text{rank}(F) = 2$, there are exactly two elements H such that $F < H < G$.

(b) (Strong connectivity): Suppose $F < G$ and $\text{rank}(G) - \text{rank}(F) \geq 3$. Then the poset interval $(F, G) := \{H : F < H < G\}$ has a connected Hasse diagram.

From now on, we will understand 'polytope' to mean 'abstract polytope'.

Each element F of rank $n-1$ in an n-polytope induces an $(n-1)$-polytope; namely, the poset $[F_{-1}, F] := \{H : F_{-1} \leq H \leq F\}$. We refer to this as a *facet* of $\mathcal{P}$. (In some contexts, 'facet' can also refer to the element F itself, but at present we will not need to refer to those elements separately.) Similarly, if v is an element of rank 0, then the *vertex-figure at v* is the poset $[v, F_n]$, which is also an $(n-1)$-polytope. Finally, a *medial section* of $\mathcal{P}$ is a facet of a vertex-figure of $\mathcal{P}$ (or indeed, a vertex-figure of a facet). Note that in the literature on polytopes, the interval $[F, G]$ is usually written G/F and called a *section* of the polytope.

The maximal chains of a polytope are referred to as *flags*. If two flags differ only in their face of rank i, then we say that the flags are i-adjacent, or simply adjacent. For every $i \in \{0, \ldots, n-1\}$, each flag has a unique i-adjacent flag.

If $\mathcal{P}$ and $\mathcal{Q}$ are polytopes of the same rank, we say that $\mathcal{P}$ *covers* $\mathcal{Q}$ if there is a function $\pi : \mathcal{P} \to \mathcal{Q}$ that preserves the partial order, the rank of each face, and such that whenever two flags of $\mathcal{P}$ are i-adjacent, so are their images under π. Note that, though we do not require surjectivity in the definition, it is in fact automatic from the conditions we require.

An *automorphism* of $\mathcal{P}$ is a bijection from $\mathcal{P}$ to itself that preserves rank and the partial order. We denote the automorphism group of $\mathcal{P}$ by $\Gamma(\mathcal{P})$. There is a natural action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$, and if this action is transitive, then $\mathcal{P}$ is called *regular*. In this case, the automorphism group of $\mathcal{P}$ is a quotient of the string Coxeter group

$$W_n = \langle r_0, \ldots, r_{n-1} \mid r_i^2 = 1 \text{ for all } i, (r_i r_j)^2 = 1 \text{ for all } |i - j| > 1 \rangle.$$

Furthermore, we can reconstruct a regular polytope $\mathcal{P}$ from the epimorphism $\pi : W_n \to \Gamma(\mathcal{P})$ as a coset geometry. For $i \in \{0, \dots, n-1\}$, let $\rho_i = \pi(r_i)$. Then we define the i-faces to be cosets of

$$G_i := \langle \rho_j \mid j \neq i \rangle$$

and where we define the partial order by $G_i\alpha \leq G_j\beta$ if and only if $i \leq j$ and $G_i\alpha \cap G_j\beta \neq \emptyset$. A regular polytope is called *orientable* (or *directly regular*) if every relator in $\Gamma(\mathcal{P})$ has even length in the generators r_i. In this case, the *rotation subgroup* of the automorphism group, which consists of the words of even length in $\Gamma(\mathcal{P})$, is a subgroup of index 2 in $\Gamma(\mathcal{P})$. We denote this group by $\Gamma^+(\mathcal{P})$.

A polytope $\mathcal{P}$ is *chiral* if the action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$ has two orbits such that adjacent flags always lie in different orbits. The facets of a chiral polytope are all isomorphic to some polytope $\mathcal{K}$ that is either chiral or directly regular, and the same is true of the vertex-figures. The facets of the facets (and vertex-figures of the vertex-figures) must be directly regular (see [14, Prop. 9]). When working with chiral polytopes, we assume that we have chosen a *base flag*; the form of the automorphism group for a chiral polytope depends on the choice of base flag.

2.2 String C^+-groups

For each $n \geq 1$, let

$$W_n^+ = \langle s_1, \dots, s_{n-1} \mid (s_i \cdots s_j)^2 = 1 \text{ for all } i < j \rangle.$$

Consider a group $\Gamma = W_n^+/M = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$, where each σ_i is the image of s_i under the natural projection. We define:

$$\Gamma_0 = \langle \sigma_2, \dots, \sigma_{n-1} \rangle$$

$$\Gamma_{n-1} = \langle \sigma_1, \dots, \sigma_{n-2} \rangle$$

$$\Gamma_{0,n-1} = \langle \sigma_2, \dots, \sigma_{n-2} \rangle.$$

The group Γ is called a *string C^+-group* if it satisfies a certain *intersection condition* (see [14, Prop. 7]), but the following recursive definition is equivalent (see for example [4, Proposition 3.13]).

Definition 2.1. *$\Gamma = W_n^+/M$ is a string C^+-group if:*

- *$n = 1$ and $|\Gamma| > 1$, or*
- *Γ_0 and Γ_{n-1} are both string C^+-groups and $\Gamma_0 \cap \Gamma_{n-1} = \Gamma_{0,n-1}$.*

The group W_n^+ has an outer automorphism that sends each s_i to $\overline{s_i}$ defined by:

$$\overline{s_1} = s_1^{-1}$$

$$\overline{s_2} = s_1^2 s_2$$

$$\overline{s_i} = s_i \text{ for } i > 2.$$

Indeed, we can identify W_n^+ with the even-word subgroup of W_n by identifying each s_i with $r_{i-1}r_i$, and the outer automorphism of W_n^+ is just conjugation by r_0 in W_n. Given a quotient $\Gamma = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ of W_n^+, we define

$$\overline{\Gamma} = \langle \overline{\sigma_1}, \dots, \overline{\sigma_{n-1}} \rangle,$$

where $\overline{\sigma_i}$ is the image of $\overline{s_i}$. Similarly, if $M \leq W_n^+$, then we define

$$\overline{M} = \{\overline{g} \mid g \in M\}.$$

The automorphism group of a chiral polytope is a string C^+-group. Similarly, the rotation subgroup of a directly regular polytope is a string C^+-group. Furthermore, we can build a polytope from a string C^+-group, and the result is either chiral or directly regular. In fact, if Γ is a string C^+-group, then it is the automorphism group of a chiral polytope if and only if there is no automorphism of Γ mapping each σ_i to $\overline{\sigma_i}$ (see [14, Thm. 1(c)]).

More generally, if Γ is a quotient of W_n^+ but not a string C^+-group, then we may nevertheless build something from it that will be somewhat like a polytope, such that the notions of regularity and chirality still make sense.

2.3 Smallest regular covers and mixing

If $G = \langle g_1, \dots, g_n \rangle$ and $H = \langle h_1, \dots, h_n \rangle$, then their *mix* is defined as

$$G \diamond H = \langle (g_1, h_1), \dots, (g_n, h_n) \rangle \leq G \times H.$$

If $\mathcal{P}$ and $\mathcal{Q}$ are directly regular n-polytopes, then we define $\mathcal{P} \diamond \mathcal{Q}$ to be the poset obtained as an appropriate coset geometry of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$. If $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ is a string C^+-group, then $\mathcal{P} \diamond \mathcal{Q}$ is a directly regular n-polytope, and is indeed the unique minimal regular n-polytope that covers both $\mathcal{P}$ and $\mathcal{Q}$. See [3] for more details.

Every chiral polytope has a unique smallest regular cover, which may or may not be a polytope (see [12, Prop. 4.1] and the remark before it.) If $\mathcal{P}$ is a chiral polytope with automorphism group Γ, then the rotation subgroup of its smallest regular cover $\mathcal{R}$ is $\Lambda = \Gamma \diamond \overline{\Gamma}$. Then $\mathcal{R}$ will be a polytope if and only if Λ is a string C^+-group.

Our goal is to characterize those chiral polytopes $\mathcal{P}$ whose smallest regular cover is a polytope. We recall the following:

Proposition 2.2. *([11, Cor. 7.5]) If $\mathcal{P}$ is a chiral polytope with regular facets or vertex-figures, then its smallest regular cover is a polytope.*

Thus, it suffices to concentrate on the case where $\mathcal{P}$ has chiral facets and vertex-figures. We start with the following simple observation:

Proposition 2.3. *Let $\mathcal{P}$ be a chiral polytope with chiral facets and vertex-figures, with automorphism group Γ, and let $\Lambda = \Gamma \diamond \overline{\Gamma}$. Then the smallest regular cover of $\mathcal{P}$ is a polytope if and only if $\Lambda_0 \cap \Lambda_{n-1} \leq \Lambda_{0,n-1}$.*

Proof. Since the facets of the facets of a chiral polytope are regular, the facets of $\mathcal{P}$ are chiral polytopes with regular facets. Thus, by Proposition 2.2, their smallest regular cover is a polytope. The automorphism group of this smallest regular cover is $\Gamma_{n-1} \diamond \overline{\Gamma_{n-1}}$, and so it is a string C^+-group. In fact, this just says that Λ_{n-1} is a string C^+-group. A dual argument shows that Λ_0 is a string C^+-group. Then it follows that Λ is a string C^+-group if and only if $\Lambda_0 \cap \Lambda_{n-1} = \Lambda_{0,n-1}$, and the inclusion $\Lambda_{0,n-1} \leq \Lambda_0 \cap \Lambda_{n-1}$ is obvious.. □

Finally, let us recall the notion of the chirality group of a polytope, as defined in [1]. If $\mathcal{P}$ is a chiral polytope with automorphism group Γ (relative to a chosen base flag), then Γ is naturally isomorphic to W_n^+/M for some normal subgroup M of W_n^+. Then the *chirality group* of $\mathcal{P}$, denoted $X(\mathcal{P})$, is the kernel of the natural projection $\pi : W_n^+/M \to W_n^+/M\overline{M}$.

Proposition 2.4. *Let $\mathcal{P}$ be a chiral polytope with automorphism group Γ, and let $\Lambda = \Gamma \diamond \overline{\Gamma}$. Then*

$$X(\mathcal{P}) = \{\gamma \in \Gamma \mid (\gamma, 1) \in \Lambda\}.$$

Proof. Note that $(\gamma, 1) \in \Lambda$ if and only if there is some $g = s_{i_1} \cdots s_{i_k} \in W_n^+$ such that g projects to γ in W_n^+/M and g projects to 1 in $W_n^+/\overline{M}$. In other words, $g \in \overline{M}$ and thus $g \in M\overline{M}$. Thus, projecting g to γ in W_n^+/M and then projecting γ to $W_n^+/M\overline{M}$ must yield 1, and so $(\gamma, 1) \in \Lambda$ if and only if $\gamma \in X(\mathcal{P})$. □

Note that if $\mathcal{P}$ has facets isomorphic to $\mathcal{K}$, then $X(\mathcal{K})$ naturally embeds into $X(\mathcal{P})$ as:

$$X(\mathcal{K}) = \{\gamma \in \Gamma_{n-1} \mid (\gamma, 1) \in \Lambda_{n-1}\}.$$

Similarly, if the vertex-figures are isomorphic to $\mathcal{L}$, then

$$X(\mathcal{L}) = \{\gamma \in \Gamma_0 \mid (\gamma, 1) \in \Lambda_0\}.$$

Finally, if the medial sections are isomorphic to $\mathcal{M}$, then

$$X(\mathcal{M}) = \{\gamma \in \Gamma_{0,n-1} \mid (\gamma, 1) \in \Lambda_{0,n-1}\}.$$

3 Main result

We are now ready for the main result.

Theorem 3.1. *Let $\mathcal{P}$ be a chiral polytope with facets isomorphic to $\mathcal{K}$, vertex-figures isomorphic to $\mathcal{L}$, and medial sections isomorphic to $\mathcal{M}$. Then the smallest regular cover of $\mathcal{P}$ is a polytope if and only if $X(\mathcal{K}) \cap X(\mathcal{L}) \leq X(\mathcal{M})$.*

Proof. First, note that

$$\begin{aligned} X(\mathcal{K}) \cap X(\mathcal{L}) &= \{\gamma \in \Gamma_{n-1} \mid (\gamma, 1) \in \Lambda_{n-1}\} \cap \{\gamma \in \Gamma_0 \mid (\gamma, 1) \in \Lambda_0\} \\ &= \{\gamma \in \Gamma_0 \cap \Gamma_{n-1} \mid (\gamma, 1) \in \Lambda_0 \cap \Lambda_{n-1}\} \\ &= \{\gamma \in \Gamma_{0,n-1} \mid (\gamma, 1) \in \Lambda_0 \cap \Lambda_{n-1}\}, \end{aligned}$$

where the last equality follows since Γ is a string C^+-group. Recall that

$$X(\mathcal{M}) = \{\gamma \in \Gamma_{0,n-1} \mid (\gamma, 1) \in \Lambda_{0,n-1}\}.$$

Thus, if $\Lambda_0 \cap \Lambda_{n-1} \leq \Lambda_{0,n-1}$, then $X(\mathcal{K}) \cap X(\mathcal{L}) \leq X(\mathcal{M})$. In light of Proposition 2.3, it remains to prove the converse.

Suppose that $(\gamma, \beta) \in \Lambda_0 \cap \Lambda_{n-1}$. In particular, $\beta \in \overline{\Gamma}_0 \cap \overline{\Gamma}_{n-1}$, and since $\overline{\Gamma}$ is a string C^+-group, it follows that $\beta \in \overline{\Gamma}_{0,n-1}$. Then there is an element α of $\Lambda_{0,n-1}$ such that $(\gamma, \beta)\alpha = (\gamma', 1)$ for some γ'. Furthermore, $(\gamma', 1) \in \Lambda_0 \cap \Lambda_{n-1}$, from which it follows that $\gamma' \in \Gamma_{0,n-1}$. Thus $(\gamma', 1) \in X(\mathcal{K}) \cap X(\mathcal{L})$ which, by assumption, is contained in $X(\mathcal{M})$. So $(\gamma', 1) \in \Lambda_{0,n-1}$, and since $\alpha \in \Lambda_{0,n-1}$, so is $(\gamma', 1)\alpha^{-1} = (\gamma, \beta)$, completing the proof. □

The beauty of Theorem 3.1 is that it does not depend on $\mathcal{P}$ per se – merely on the type of its facets and vertex-figures. Let us consider several examples and corollaries.

Remark 3.2. *Note that Theorem 3.1 does not require that the facets and vertex-figures of $\mathcal{P}$ both be chiral. If, for example, the facets $\mathcal{K}$ are regular, then $X(\mathcal{K})$ is trivial, and so the condition of the theorem is trivially satisfied, agreeing with Proposition 2.2.*

Corollary 3.3. *Suppose $\mathcal{P}$ is a chiral polytope with automorphism group Γ, facets isomorphic to $\mathcal{K}$, and vertex-figures isomorphic to $\mathcal{L}$. If the medial sections of $\mathcal{P}$ are regular, then the smallest regular cover of $\mathcal{P}$ is a polytope if and only if $X(\mathcal{K}) \cap X(\mathcal{L})$ (viewed as subgroups of Γ) has trivial intersection with $\Gamma_{0,n-1}$.*

Corollary 3.4. *If $\mathcal{P}$ is a chiral 4-polytope with automorphism group Γ, facets isomorphic to $\mathcal{K}$, and vertex-figures isomorphic to $\mathcal{L}$, then the smallest regular cover of $\mathcal{P}$ is a polytope if and only if no power of σ_2 is contained in $X(\mathcal{K}) \cap X(\mathcal{L})$.*

Example 3.5. *The five families of chiral 4-polytopes with chiral facets (and vertex-figures) in [7, Table 4] have a power of σ_2 in $X(\mathcal{K}) \cap X(\mathcal{L})$ (see [7, Prop. 11]), and so the smallest regular cover of each one is not a polytope.*

A chiral polytope $\mathcal{P}$ is called *totally chiral* if $X(\mathcal{P}) = \Gamma$ (see [1]). The following result is immediate.

Proposition 3.6. *If $\mathcal{P}$ has totally chiral facets and vertex-figures, then its smallest regular cover is a polytope if and only if it has totally chiral medial sections.*

Finally, we note that Theorem 3.1 guarantees that certain choices of facets for a chiral polytope will guarantee that the smallest regular cover is a polytope.

Corollary 3.7. *Suppose that $\mathcal{K}$ is a chiral polytope with automorphism group Γ. If $X(\mathcal{K})$ has trivial intersection with Γ_0, then the smallest regular cover of every chiral polytope with facets isomorphic to $\mathcal{K}$ is a polytope.*

Example 3.8. *The chiral polyhedra $\{4,4\}_{(b,c)}$ with $bc(b-c) \neq 0$ all have chirality groups that trivially intersect $\langle \sigma_2 \rangle$. (Indeed, all such polyhedra cover the regular map $\{4,4\}_{(1,0)}$, and so no power of σ_2 can be present in the chirality group.) Thus, all chiral polytopes with such facets have a smallest regular cover that is a polytope.*

4 Minimal regular covers

A regular polytope $\mathcal{R}$ is said to be a *minimal regular cover* of the polytope $\mathcal{P}$ if the only regular cover of $\mathcal{P}$ that is covered by $\mathcal{R}$ is $\mathcal{R}$ itself. Whenever the smallest regular cover of $\mathcal{P}$ is a polytope, then this is the unique minimal regular cover of $\mathcal{P}$ [11, Prop. 3.16]. Otherwise, a polytope may have multiple minimal regular covers; for example, see the Tomotope [10]. Indeed, it seems likely that every polytope whose smallest regular cover is not a polytope has multiple minimal regular covers. Problem 4 in [13] asks for a characterization of chiral polytopes with a unique minimal regular cover.

Let us look at an example inspired by Example 3.5. Consider the chiral polytope $\mathcal{P}$ whose automorphism group is

$$\Gamma = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^6 = \sigma_2^9 = \sigma_3^6 = 1, (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1, \sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^4, \sigma_3^2\sigma_2 = \sigma_2^4\sigma_3^2 \rangle.$$

This is a polytope with 648 flags, and its smallest regular cover $\mathcal{R}$, with 1944 flags, is not a polytope. Clearly, any regular polytope $\mathcal{Q}$ that covers $\mathcal{R}$ such that $|\mathcal{Q}|/|\mathcal{R}|$ is a prime number will be a minimal regular cover of $\mathcal{P}$.

Now, let $\mathcal{K}$ be a facet of $\mathcal{R}$, and let $\mathcal{T}$ be its *trivial extension*, which has rotation subgroup

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^6 = \sigma_2^9 = \sigma_3^2 = 1, (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1, (\sigma_1\sigma_2^{-2})^2 = \sigma_1\sigma_2^2\sigma_1(\sigma_1\sigma_2^{-1}\sigma_1)^2 = 1 \rangle.$$

Let $\mathcal{Q} = \mathcal{R} \diamond \mathcal{T}$. Using the GAP package RAMP [6, 8], we confirmed that $\mathcal{Q}$ is a regular polytope with 5832 flags, making it a 3-fold cover of $\mathcal{R}$ and thus a minimal regular cover of $\mathcal{P}$. Similarly, if we mix $\mathcal{R}$ with the dual of $\mathcal{T}$, which has group

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^9 = \sigma_3^6 = 1, (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1, (\sigma_3^{-1}\sigma_2^2)^2 = \sigma_2\sigma_3^2\sigma_2(\sigma_3^{-1}\sigma_2\sigma_3^{-1})^2 = 1 \rangle,$$

then we obtain another 3-fold cover of $\mathcal{R}$, and these two covers are non-isomorphic. So $\mathcal{P}$ does not have a unique minimal regular cover. We conjecture that the situation is similar with all of the chiral polytopes mentioned in Example 3.5.

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