Variance groups and the structure of mixed polytopes

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Abstract

The natural mixing construction for abstract polytopes provides a way to build a minimal common cover of two regular or chiral polytopes. With the help of the chirality group of a polytope, it is often possible to determine when the mix of two chiral polytopes is still chiral. By generalizing the chirality group to a whole family of variance groups, we can explicitly describe the structure of the mix of two polytopes. We are also able to determine when the mix of two polytopes is invariant under other external symmetries, such as duality and Petrie duality.

Key Words: abstract regular polytope, chiral polytope, self-dual polytope, chiral map, Petrie dual, external symmetry.

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1 Introduction

The study of abstract polytopes, together with the study of maps on surfaces, is a vibrant area of current research. These fields bring together group theory, geometry, and combinatorics in a satisfying way, providing many fascinating structures to study. As in the classical theory of convex polytopes, the regular polytopes are particularly interesting. In our context, a polytope is regular if its automorphism group acts transitively on the flags. Also important are the chiral polytopes, whose defining features are that there are two flag orbits under the action of the automorphism group, and that flags that differ in only a single element lie in different orbits. Such polytopes occur in two mirror-image forms, but they have full rotational symmetry.

In addition to internal symmetries, which are represented by polytope automorphisms, there are a number of interesting external symmetries. Some, such as duality, have their roots in the study of convex polytopes. Others, such as Petrie duality, come more naturally from the study of maps on surfaces. Even the symmetry between a chiral polytope and its mirror image may be viewed as an external symmetry. Most of the work on external symmetries has focused on polyhedra and maps on surfaces (see [9, 13, 14, 18]), though some work has been done with polytopes in higher rank as well (see [11]).

Polytopes have a natural mixing construction, analogous to the join of two maps or hypermaps [3]. This construction lets us build the minimal common cover of two regular or chiral polytopes. Unlike joining maps, there is a significant hurdle when mixing polytopes; namely, there is no guarantee that the mix of two polytopes is itself a polytope. In some cases, we are able to determine whether the mix of two polytopes is polytopal based on simple combinatorial data.

By mixing a polytope with its images under an external symmetry or a group of symmetries, we can construct a polytope (or a slightly more general structure) that is invariant under that symmetry or symmetries. For example, we can build polytopes that are self-dual, self-Petrie, or both. Our goal then becomes determining the full structure of the mixed polytope, including its combinatorial data and its automorphism group.

Sometimes we are more interested in constructing polytopes that are *not* invariant under a given external symmetry. For example, we would like to know when the mix of two chiral polytopes is still chiral. By using the chirality group, which measures the degree to which a polytope is chiral, we can often make this determination (see [1, 2, 7]). The chirality group generalizes nicely, making it possible to measure how far a polytope is from being invariant under any external symmetry.

We start by giving background information on regular and chiral polytopes in Section 2. In Section 3, we introduce the mixing construction for polytopes and investigate the structure of mixed polytopes. Then we develop the theory of external symmetries in Section 4. The main result is Theorem 4.9, which uses a generalization of the chirality group to determine when the mix of two polytopes is invariant under an external symmetry. We then provide several consequences and examples.

2 Polytopes

General background information on abstract polytopes can be found in [16, Chs. 2, 3], and information on chiral polytopes specifically can be found in [19]. Here we review the concepts essential for this paper.

2.1 Definition of a polytope

Let $\mathcal{P}$ be a ranked partially ordered set whose elements will be called *faces*. The faces of $\mathcal{P}$ will range in rank from -1 to n, and a face of rank j is called a *j-face*. The 0-faces, 1-faces, and $(n-1)$-faces are also called *vertices*, *edges*, and *facets*, respectively. A *flag* of $\mathcal{P}$ is a maximal chain. We say that two flags are *adjacent* if they differ in exactly one face, and that they are *j-adjacent* if they differ only in their j-face. If F and G are faces of $\mathcal{P}$ such that $F \leq G$, then the *section* G/F consists of those faces H such that $F \leq H \leq G$.

We say that $\mathcal{P}$ is an *(abstract) polytope of rank* n, also called an n*-polytope*, if it satisfies the following four properties:

(a) There is a unique greatest face F_n of rank n and a unique least face F_{-1} of rank -1.

(b) Each flag of $\mathcal{P}$ has $n+2$ faces.

(c) $\mathcal{P}$ is *strongly flag-connected*, meaning that if Φ and Ψ are two flags of $\mathcal{P}$, then there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that for $i = 0, \ldots, k-1$, the flags Φ_i and Φ_{i+1} are adjacent, and each Φ_i contains $\Phi \cap \Psi$.

(d) (Diamond condition): Whenever $F < G$, where F is a $(j-1)$-face and G is a $(j+1)$-face for some j, then there are exactly two j-faces H with $F < H < G$.

Note that due to the diamond condition, any flag Φ has a unique j-adjacent flag (denoted Φ^j) for each $j = 0, 1, \ldots, n-1$.

If F is a j-face and G is a k-face of a polytope with $F \leq G$, then the section G/F is a $(k-j-1)$-polytope itself. We can identify a face F with the section F/F_{-1}, since if F is a j-face, then F/F_{-1} is a j-polytope. We call the section F_n/F the *co-face at* F; the co-face at a vertex is also called a *vertex-figure*.

We sometimes need to work with *pre-polytopes*, which are ranked partially ordered sets that satisfy the first, second, and fourth property above, but not necessarily the third. In this paper, all of the pre-polytopes we encounter will be *flag-connected*, meaning that if Φ and Ψ are two flags, there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that for $i = 0, \ldots, k-1$, the flags Φ_i and Φ_{i+1} are adjacent (but we do not require each flag to contain $\Phi \cap \Psi$). When working with pre-polytopes, we apply all the same terminology as with polytopes.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes (or flag-connected pre-polytopes) of the same rank. A function $\gamma : \mathcal{P} \to \mathcal{Q}$ is called a *covering* if it preserves incidence of faces, ranks of faces, and adjacency of flags; then γ is necessarily surjective, by the flag-connectedness of $\mathcal{Q}$. We say that $\mathcal{P}$ *covers* $\mathcal{Q}$ if there exists a covering $\gamma : \mathcal{P} \to \mathcal{Q}$.

2.2 Regularity

For polytopes $\mathcal{P}$ and $\mathcal{Q}$, an *isomorphism* from $\mathcal{P}$ to $\mathcal{Q}$ is an incidence- and rank-preserving bijection on the set of faces. An isomorphism from $\mathcal{P}$ to itself is an *automorphism* of $\mathcal{P}$, and the group of all automorphisms of $\mathcal{P}$ is denoted $\Gamma(\mathcal{P})$. We say that $\mathcal{P}$ is *regular* if the natural action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$ is transitive. For convex polytopes, this definition is equivalent to any of the usual definitions of regularity (see [16, Sect. 1B]).

Given a regular polytope $\mathcal{P}$, fix a *base flag* Φ. Then the automorphism group $\Gamma(\mathcal{P})$ is generated by the *abstract reflections* $\rho_0, \ldots, \rho_{n-1}$, where ρ_i maps Φ to the unique flag Φ^i that is i-adjacent to Φ. These generators satisfy $\rho_i^2 = \epsilon$ for all i, and $(\rho_i \rho_j)^2 = \epsilon$ for all i and j such that $|i-j| \geq 2$. We say that $\mathcal{P}$ has *(Schläfli) type* $\{p_1, \ldots, p_{n-1}\}$ if for each $i = 1, \ldots, n-1$ the order of $\rho_{i-1}\rho_i$ is p_i (with $2 \leq p_i \leq \infty$).

For $I \subseteq \{0, 1, \ldots, n-1\}$ and a group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, we define $\Gamma_I := \langle \rho_i \mid i \in I \rangle$. The strong flag-connectivity of regular polytopes induces the following *intersection condition* in the group:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \quad \text{for } I, J \subseteq \{0, \ldots, n-1\}. \tag{1}$$

In general, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a group such that each ρ_i has order 2 and such that $(\rho_i \rho_j)^2 = \epsilon$ whenever $|i-j| \geq 2$, then we say that Γ is a *string group generated by involutions* (or *sggi*). If Γ also satisfies the intersection condition (1) given above, then we call Γ a *string C-group*. There is a natural way of building a regular polytope $\mathcal{P}(\Gamma)$ from a string C-group Γ such that $\Gamma(\mathcal{P}(\Gamma)) = \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) = \mathcal{P}$ (see [16, Ch. 2E]). In particular, the i-faces of $\mathcal{P}(\Gamma)$ are taken to be the cosets of

$$\Gamma_i := \langle \rho_j \mid j \neq i \rangle,$$

where $\Gamma_i \varphi \leq \Gamma_j \psi$ if and only if $i \leq j$ and $\Gamma_i \varphi \cap \Gamma_j \psi \neq \emptyset$. This construction is also easily applied to any sggi (not just string C-groups), but in that case, the resulting poset is not necessarily a polytope.

If $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes, their automorphism groups are both quotients of the Coxeter group

$$W_n := [\infty, \ldots, \infty] = \langle \rho_0, \ldots, \rho_{n-1} \mid \rho_0^2 = \cdots = \rho_{n-1}^2 = \epsilon, (\rho_i \rho_j)^2 = \epsilon \text{ when } |i-j| \geq 2 \rangle. \tag{2}$$

Therefore there are normal subgroups M and K of W_n such that $\Gamma(\mathcal{P}) = W_n/M$ and $\Gamma(\mathcal{Q}) = W_n/K$. Then $\mathcal{P}$ covers $\mathcal{Q}$ if and only if $M \leq K$.

2.3 Direct Regularity and Chirality

If $\mathcal{P}$ is a regular polytope with automorphism group $\Gamma(\mathcal{P})$ generated by $\rho_0, \ldots, \rho_{n-1}$, then the *abstract rotations*

$$\sigma_i := \rho_{i-1} \rho_i \ (i = 1, \ldots, n-1)$$

generate the *rotation subgroup* $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$, which has index at most 2. We say that $\mathcal{P}$ is *directly regular* if this index is 2. This is essentially an orientability condition; for example, the directly regular polyhedra correspond to orientable regular maps. The convex regular polytopes are all directly regular.

We say that an n-polytope $\mathcal{P}$ is *chiral* if the action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$ has two orbits such that adjacent flags are always in distinct orbits. For convenience, we define $\Gamma^+(\mathcal{P}) := \Gamma(\mathcal{P})$ whenever $\mathcal{P}$ is chiral. Given a chiral polytope $\mathcal{P}$ with base flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$, the automorphism group $\Gamma^+(\mathcal{P})$ is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i acts on Φ the same way that $\rho_{i-1}\rho_i$ acts on the base flag of a regular polytope. That is, σ_i sends Φ to $(\Phi^i)^{i-1}$ (which is usually denoted $\Phi^{i,i-1}$). For $i < j$, the product $\sigma_i \cdots \sigma_j$ is an involution. In analogy to regular polytopes, if the order of each σ_i is p_i, we say that the *type* of $\mathcal{P}$ is $\{p_1, \ldots, p_{n-1}\}$.

The automorphism groups of chiral polytopes and the rotation groups of directly regular polytopes satisfy an intersection property analogous to that for string C-groups. Let $\Gamma^+ :=$

$\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ be the automorphism group of a chiral polytope or the rotation subgroup of a directly regular polytope $\mathcal{P}$. For $1 \le i < j \le n-1$ define $\tau_{i,j} := \sigma_i \cdots \sigma_j$. By convention, we also define $\tau_{i,i} = \sigma_i$, and for $0 \le i \le n$, we define $\tau_{0,i} = \tau_{i,n} = \epsilon$. For $I \subseteq \{0, \ldots, n-1\}$, set

$$\Gamma_I^+ := \langle \tau_{i,j} \mid i \le j \text{ and } i-1, j \in I \rangle.$$

Then the *intersection property* for Γ^+ is given by:

$$\Gamma_I^+ \cap \Gamma_J^+ = \Gamma_{I \cap J}^+ \quad \text{for } I, J \subseteq \{0, \ldots, n-1\}. \tag{3}$$

If Γ^+ is a group generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ such that $(\sigma_i \cdots \sigma_j)^2 = \epsilon$ for $i < j$, and if Γ^+ satisfies the intersection property (3) above, then Γ^+ is either the automorphism group of a chiral n-polytope or the rotation subgroup of a directly regular polytope. In particular, it is the rotation subgroup of a directly regular polytope if and only if there is a group automorphism of Γ^+ that sends σ_1 to σ_1^{-1}, σ_2 to $\sigma_1^2\sigma_2$, and fixes every other generator.

Suppose $\mathcal{P}$ is a chiral polytope with base flag Φ and with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. Let $\overline{\mathcal{P}}$ be the chiral polytope with the same underlying face-set as $\mathcal{P}$, but with base flag Φ^0. Then $\Gamma^+(\overline{\mathcal{P}}) = \langle \sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1} \rangle$. We call $\overline{\mathcal{P}}$ the *enantiomorphic form* or *mirror image* of $\mathcal{P}$. Though $\mathcal{P} \simeq \overline{\mathcal{P}}$, there is no automorphism of $\mathcal{P}$ that takes Φ to Φ^0.

Let $\Gamma^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, and let w be a word in the free group on these generators. We define the *enantiomorphic* (or *mirror image*) word $\overline{w}$ of w to be the word obtained from w by replacing every occurrence of σ_1 by σ_1^{-1} and σ_2 by $\sigma_1^2\sigma_2$, while keeping all σ_j with $j \ge 3$ unchanged. Then if Γ^+ is the rotation subgroup of a directly regular polytope, the elements of Γ^+ corresponding to w and $\overline{w}$ are conjugate in the full group Γ. On the other hand, if Γ^+ is the automorphism group of a chiral polytope, then w and $\overline{w}$ need not even have the same period. Note that $\overline{\overline{w}} = w$ for all words w.

The sections of a regular polytope are again regular, and the sections of a chiral polytope are either directly regular or chiral. Furthermore, for a chiral n-polytope, all the $(n-2)$-faces and all the co-faces at edges must be directly regular [19]. As a consequence, if $\mathcal{P}$ is a chiral polytope, it may be possible to extend it to a chiral polytope with facets isomorphic to $\mathcal{P}$, but it will then be impossible to extend that polytope once more to a chiral polytope.

Chiral polytopes only exist in ranks 3 and higher. The simplest examples are the torus maps $\{4,4\}_{(b,c)}$, $\{3,6\}_{(b,c)}$ and $\{6,3\}_{(b,c)}$, with $b, c \neq 0$ and $b \neq c$ (see [5]). These give rise to chiral 4-polytopes having toroidal maps as facets and/or vertex-figures. More examples of chiral 4- and 5-polytopes can be found in [4].

If a regular or chiral n-polytope $\mathcal{P}$ has facets $\mathcal{K}$ and vertex-figures $\mathcal{L}$, we say that $\mathcal{P}$ is of *type* $\{\mathcal{K}, \mathcal{L}\}$. If $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$ and it covers every other polytope of the same type, then we say that $\mathcal{P}$ is the *universal polytope of type* $\{\mathcal{K}, \mathcal{L}\}$, and we simply denote it by $\{\mathcal{K}, \mathcal{L}\}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are chiral or directly regular n-polytopes, their rotation groups are both quotients of

$$W_n^+ := [\infty, \ldots, \infty]^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \mid (\sigma_i \cdots \sigma_j)^2 = \epsilon \text{ for } 1 \le i < j \le n-1 \rangle.$$

Therefore there are normal subgroups M and K of W_n^+ such that $\Gamma^+(\mathcal{P}) = W_n^+/M$ and $\Gamma^+(\mathcal{Q}) = W_n^+/K$. Then $\mathcal{P}$ covers $\mathcal{Q}$ if and only if $M \le K$.

Let $\mathcal{P}$ be a chiral or directly regular polytope with $\Gamma^+(\mathcal{P}) = W_n^+/M$. We define

$$\overline{M} = \{\overline{w} \mid w \in M\}.$$

Note that $\overline{M} = \rho_0 M \rho_0$, where as before, ρ_0 is the first standard generator of W_n. If $\overline{M} = M$, then $\mathcal{P}$ is directly regular. Otherwise, $\mathcal{P}$ is chiral, and $\Gamma^+(\overline{\mathcal{P}}) = W_n^+/\overline{M}$.

2.4 Duality and Petrie duality

For any polytope $\mathcal{P}$, we obtain the *dual of* $\mathcal{P}$ (denoted $\mathcal{P}^\delta$) by simply reversing the partial order. A *duality* from $\mathcal{P}$ to $\mathcal{Q}$ is an anti-isomorphism, that is, a bijection δ between the face sets such that $F < G$ in $\mathcal{P}$ if and only if $\delta(F) > \delta(G)$ in $\mathcal{Q}$. If a polytope is isomorphic to its dual, then it is called *self-dual*.

If $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$, then $\mathcal{P}^\delta$ is of type $\{\mathcal{L}^\delta, \mathcal{K}^\delta\}$. Therefore, in order for $\mathcal{P}$ to be self-dual, it is necessary (but not sufficient) that $\mathcal{K}$ be isomorphic to $\mathcal{L}^\delta$ (in which case it is also true that $\mathcal{K}^\delta$ is isomorphic to $\mathcal{L}$).

A self-dual regular polytope always possesses a duality that fixes the base flag. For chiral polytopes, this may not be the case. If a self-dual chiral polytope $\mathcal{P}$ possesses a duality that sends the base flag to another flag in the same orbit (but reversing its direction), then there is a duality that fixes the base flag, and we say that $\mathcal{P}$ is *properly self-dual* [12]. In this case, the groups $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{P}^\delta)$ have identical presentations. If a self-dual chiral polytope has no duality that fixes the base flag, then every duality sends the base flag to a flag in the other orbit, and $\mathcal{P}$ is said to be *improperly self-dual*. In this case, the groups $\Gamma^+(\overline{\mathcal{P}})$ and $\Gamma^+(\mathcal{P}^\delta)$ have identical presentations instead.

If $\mathcal{P}$ is a regular polytope with $\Gamma(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$, then the group of $\mathcal{P}^\delta$ is $\Gamma(\mathcal{P}^\delta) = \langle \rho_0', \dots, \rho_{n-1}' \rangle$, where $\rho_i' = \rho_{n-1-i}$. If $\mathcal{P}$ is a directly regular or chiral polytope with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$, then the rotation group of $\mathcal{P}^\delta$ is $\Gamma^+(\mathcal{P}^\delta) = \langle \sigma_1', \dots, \sigma_{n-1}' \rangle$, where $\sigma_i' = \sigma_{n-i}^{-1}$. Equivalently, if $\Gamma^+(\mathcal{P})$ has presentation

$$\langle \sigma_1, \dots, \sigma_{n-1} \mid w_1, \dots, w_k \rangle$$

then $\Gamma^+(\mathcal{P}^\delta)$ has presentation

$$\langle \sigma_1', \dots, \sigma_{n-1}' \mid \delta(w_1), \dots, \delta(w_k) \rangle,$$

where if $w = \sigma_{i_1} \cdots \sigma_{i_j}$, then $\delta(w) = (\sigma_{n-i_1}')^{-1} \cdots (\sigma_{n-i_j}')^{-1}$.

Suppose $\mathcal{P}$ is a chiral or directly regular polytope with $\Gamma^+(\mathcal{P}) = W_n^+/M$. Then $\Gamma^+(\mathcal{P}^\delta) = W_n^+/\delta(M)$, where $\delta(M) = \{\delta(w) \mid w \in M\}$. If $\delta(M) = M$, then $\Gamma^+(\mathcal{P}) = \Gamma^+(\mathcal{P}^\delta)$, so $\mathcal{P}$ is properly self-dual.

If $\mathcal{P}$ is a chiral polytope, then $\overline{\mathcal{P}^\delta}$ is naturally isomorphic to $\overline{\mathcal{P}}^\delta$. Indeed, if w is a word in the generators $\sigma_1, \dots, \sigma_{n-1}$ of $\Gamma^+(\mathcal{P})$, then

$$\delta(\overline{w}) = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \overline{\delta(w)} (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^{-1},$$

so we see that the presentation for $\overline{\mathcal{P}^\delta}$ is equivalent to that of $\overline{\mathcal{P}}^\delta$. In particular, if $\Gamma^+(\mathcal{P}) = W_n^+/M$, then

$$\delta(\overline{M}) = \overline{\delta(M)} := \{\overline{\delta(w)} \mid w \in M\},$$

since M is a normal subgroup of W_n^+, and thus $\overline{\delta(\overline{\delta(M)})} = M$.

There is also a second duality operation that is defined on abstract polyhedra. To start with, a *Petrie polygon* of a polyhedron is a maximal edge-path such that every two successive edges lie on a common face, but no three successive edges do. Given a polyhedron $\mathcal{P}$, its *Petrie dual* $\mathcal{P}^\pi$ consists of the same vertices and edges as $\mathcal{P}$, but its faces are the Petrie polygons of $\mathcal{P}$. Taking the Petrie dual of a polyhedron also forces the old faces to be the new Petrie polygons, so that $\mathcal{P}^{\pi\pi} \simeq \mathcal{P}$. If $\mathcal{P}$ is isomorphic to $\mathcal{P}^\pi$, then we say that $\mathcal{P}$ is *self-Petrie*.

The Petrie dual of an arbitrary polyhedron need not be a polyhedron itself. In particular, a Petrie polygon may visit a single vertex multiple times, causing there to be more than two edges incident on that Petrie polygon and vertex. When $\mathcal{P}$ is regular, however, the Petrie dual is a polyhedron except in rare cases; see [16, Sect. 7B].

3 Mixing polytopes

The mixing operation on polytopes ([16], [17]) is analogous to the parallel product of groups [20], the tensor product of graphs, and the join of maps and hypermaps [3]. It gives us a natural way to find the minimal common cover of two regular or chiral polytopes. The basic method is to find the parallel product of the automorphism groups (or rotation groups) of two polytopes, and then to build a poset (usually a pre-polytope) from the resulting group. There are two main challenges. First, we want to determine how the structure of the mix depends on the two component polytopes. Second, we want to know when the mix of two polytopes is a polytope, and not just a pre-polytope. In a wide variety of cases, it is possible to easily determine the structure and polytopality of the mix.

3.1 Mixing finitely generated groups

Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$ be finitely generated groups on n generators. Then the elements $z_i := (x_i, x'_i) \in \Gamma \times \Gamma'$ (for $i = 1, \ldots, n$) generate a subgroup of $\Gamma \times \Gamma'$ that we call the *mix* of Γ and Γ', denoted $\Gamma \diamond \Gamma'$ (see [16, Ch.7A]).

If $\mathcal{P}$ and $\mathcal{Q}$ are regular polytopes, then we can mix their automorphism groups $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. The result will be an sggi, but not necessarily a string C-group. In any case, we can always build a ranked poset from $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$, and the result, which we emphasize might not be a polytope, is called the *mix of* $\mathcal{P}$ *and* $\mathcal{Q}$ and is denoted $\mathcal{P} \diamond \mathcal{Q}$. Similarly, if $\mathcal{P}$ and $\mathcal{Q}$ are chiral or directly regular, we can mix $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{Q})$ and build a poset from the resulting group. That poset is also called the mix of $\mathcal{P}$ and $\mathcal{Q}$ and denoted $\mathcal{P} \diamond \mathcal{Q}$. There is no chance for confusion, since if $\mathcal{P}$ and $\mathcal{Q}$ are both directly regular, then the poset built from $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ is the same as that built from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$.

If the facets of $\mathcal{P}$ are isomorphic to $\mathcal{K}$ and the facets of $\mathcal{Q}$ are isomorphic to $\mathcal{L}$, then the facets of $\mathcal{P} \diamond \mathcal{Q}$ are isomorphic to $\mathcal{K} \diamond \mathcal{L}$. The vertex-figures of $\mathcal{P} \diamond \mathcal{Q}$ are analogously obtained. If $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$ and $\mathcal{Q}$ is of type $\{q_1, \ldots, q_{n-1}\}$, then $\mathcal{P} \diamond \mathcal{Q}$ is of type $\{\ell_1, \ldots, \ell_{n-1}\}$, where $\ell_i = \mathrm{lcm}(p_i, q_i)$ for $i \in \{1, \ldots, n-1\}$.

In order to avoid duplication, we shall usually assume that $\mathcal{P}$ and $\mathcal{Q}$ are chiral or directly regular, and we will work with $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ instead of $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$. Most of our results are easily modified to work for $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ when necessary.

The automorphism group of a chiral or directly regular n-polytope can always be written as a natural quotient of W_n^+. The mix of two polytopes has a simple interpretation in terms of these quotients [2]:

Proposition 3.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes with $\Gamma^+(\mathcal{P}) = W_n^+/M$ and $\Gamma^+(\mathcal{Q}) = W_n^+/K$. Then $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) \simeq W_n^+/(M \cap K)$.*

Corollary 3.2. *Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ be chiral or directly regular n-polytopes. If $\mathcal{R}$ covers $\mathcal{P}$ and $\mathcal{Q}$, then it covers $\mathcal{P} \diamond \mathcal{Q}$.*

Dual to the mix is the *comix* of two groups. We define the comix $\Gamma \,\square\, \Gamma'$ to be the amalgamated free product that identifies the generators of Γ with the corresponding generators of Γ'. That is, if Γ has presentation $\langle x_1, \ldots, x_n \mid R\rangle$ and Γ' has presentation $\langle x_1', \ldots, x_n' \mid S\rangle$, then $\Gamma \,\square\, \Gamma'$ has presentation

$$\langle x_1, x_1', \ldots, x_n, x_n' \mid R, S, x_1^{-1}x_1', \ldots, x_n^{-1}x_n'\rangle.$$

Equivalently, we can just add the relations from Γ' to those of Γ, replacing each x_i' with x_i.

Just as the mix of two rotation groups has a simple description in terms of quotients of W_n^+, so does the comix of two rotation groups [6]:

Proposition 3.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes with $\Gamma^+(\mathcal{P}) = W_n^+/M$ and $\Gamma^+(\mathcal{Q}) = W_n^+/K$. Then $\Gamma^+(\mathcal{P}) \,\square\, \Gamma^+(\mathcal{Q}) \simeq W_n^+/MK$.*

The mixing and comixing operations on polytopes are commutative and associative in the sense that, for example, $\mathcal{P} \diamond \mathcal{Q}$ is naturally isomorphic to $\mathcal{Q} \diamond \mathcal{P}$. Furthermore, $\mathcal{P} \diamond \mathcal{P}$ and $\mathcal{P} \,\square\, \mathcal{P}$ are both naturally isomorphic to $\mathcal{P}$. However, even if $\mathcal{P} \simeq \mathcal{Q}$, it may be the case that $\mathcal{P} \diamond \mathcal{Q} \not\simeq \mathcal{P}$. For example, if $\mathcal{P}$ is a chiral polytope, then $\mathcal{P} \simeq \overline{\mathcal{P}}$, but $\mathcal{P} \diamond \overline{\mathcal{P}}$ is not isomorphic to $\mathcal{P}$.

3.2 Variance groups and the structure of the mix

There is a natural epimorphism from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ to $\Gamma^+(\mathcal{P})$, sending each generator (σ_i, σ_i') to σ_i. By studying the kernel of this epimorphism and the analogous epimorphism to $\Gamma^+(\mathcal{Q})$, we can determine the structure of the $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$.

Definition 3.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes. We denote the kernel of the natural epimorphism*

$$f : \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P})$$

by $X(\mathcal{Q}|\mathcal{P})$*, and we call it* the variance group of $\mathcal{Q}$ with respect to $\mathcal{P}$*. Similarly, we denote the kernel of the natural epimorphism*

$$f' : \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{Q})$$

by $X(\mathcal{P}|\mathcal{Q})$ *and we call it the variance group of* $\mathcal{P}$ *with respect to* $\mathcal{Q}$*. In other words,* $X(\mathcal{Q}|\mathcal{P})$ *consists of the elements of* $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ *of the form* (ϵ, w') *(with* $w' \in \Gamma^+(\mathcal{Q})$*), and* $X(\mathcal{P}|\mathcal{Q})$ *consists of the elements of* $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ *of the form* (w, ϵ) *(with* $w \in \Gamma^+(\mathcal{P})$*).*

By representing $\Gamma^+(\mathcal{P})$ as W_n^+/M and $\Gamma^+(\mathcal{Q})$ as W_n^+/K, we easily obtain the following:

Proposition 3.5. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be chiral or directly regular n-polytopes, with* $\Gamma^+(\mathcal{P}) = W_n^+/M$ *and* $\Gamma^+(\mathcal{Q}) = W_n^+/K$*. Then:*

(a) $X(\mathcal{P}|\mathcal{Q}) \simeq K/(M \cap K) \simeq MK/M$ *and* $X(\mathcal{Q}|\mathcal{P}) \simeq M/(M \cap K) \simeq MK/K$*.*

(b) *Let* $g : \Gamma^+(\mathcal{P}) \to \Gamma^+(\mathcal{P}) \,\Box\, \Gamma^+(\mathcal{Q})$ *and* $g' : \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P}) \,\Box\, \Gamma^+(\mathcal{Q})$ *be the natural epimorphisms. Then* $\ker g \simeq X(\mathcal{P}|\mathcal{Q})$ *and* $\ker g' \simeq X(\mathcal{Q}|\mathcal{P})$*. In particular,* $X(\mathcal{P}|\mathcal{Q})$ *and* $X(\mathcal{Q}|\mathcal{P})$ *can be viewed as normal subgroups of* $\Gamma^+(\mathcal{P})$ *and* $\Gamma^+(\mathcal{Q})$*, respectively.*

The fact that the natural epimorphisms $f' : \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{Q})$ and $g : \Gamma^+(\mathcal{P}) \to \Gamma^+(\mathcal{P}) \,\Box\, \Gamma^+(\mathcal{Q})$ have isomorphic kernels allows us to use the comix of two polytopes to derive information about the mix. The following properties are immediate:

Proposition 3.6. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be finite chiral or directly regular n-polytopes. Then:*

(a) $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ *is finite, and*

$$|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})| = |X(\mathcal{P}|\mathcal{Q})| \cdot |\Gamma^+(\mathcal{Q})| = |X(\mathcal{Q}|\mathcal{P})| \cdot |\Gamma^+(\mathcal{P})|.$$

(b) $\Gamma^+(\mathcal{P}) \,\Box\, \Gamma^+(\mathcal{Q})$ *is finite, and*

$$|\Gamma^+(\mathcal{P}) \,\Box\, \Gamma^+(\mathcal{Q})| = \frac{|\Gamma^+(\mathcal{P})|}{|X(\mathcal{P}|\mathcal{Q})|} = \frac{|\Gamma^+(\mathcal{Q})|}{|X(\mathcal{Q}|\mathcal{P})|}.$$

(c)

$$|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})| \cdot |\Gamma^+(\mathcal{P}) \,\Box\, \Gamma^+(\mathcal{Q})| = |\Gamma^+(\mathcal{P})| \cdot |\Gamma^+(\mathcal{Q})|.$$

(d)

$$\frac{|X(\mathcal{P}|\mathcal{Q})|}{|X(\mathcal{Q}|\mathcal{P})|} = \frac{|\Gamma^+(\mathcal{P})|}{|\Gamma^+(\mathcal{Q})|}$$

Intuitively speaking, the group $X(\mathcal{Q}|\mathcal{P})$ tells us something about how many elements of $\Gamma^+(\mathcal{Q})$ do not correspond to elements of $\Gamma^+(\mathcal{P})$. If $\Gamma^+(\mathcal{P})$ covers $\Gamma^+(\mathcal{Q})$, then $X(\mathcal{Q}|\mathcal{P})$ is trivial. At the other extreme, if $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$, then $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{Q})$ have trivial overlap, and $X(\mathcal{Q}|\mathcal{P}) \simeq \Gamma^+(\mathcal{Q})$.

The group $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ is a subdirect product of $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{Q})$, and we can determine its structure explicitly:

Proposition 3.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $\Gamma^+(\mathcal{Q}) = \langle \sigma'_1, \ldots, \sigma'_n \rangle$. Let $N = X(\mathcal{P}|\mathcal{Q})$ and $N' = X(\mathcal{Q}|\mathcal{P})$, and let $h : \Gamma^+(\mathcal{P})/N \to \Gamma^+(\mathcal{Q})/N'$ be the isomorphism sending $\sigma_i N$ to $\sigma'_i N'$. Then*

$$\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \{(u, v) \in \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q}) \mid h(uN) = vN'\}.$$

In particular, $X(\mathcal{P}|\mathcal{Q}) \times X(\mathcal{Q}|\mathcal{P})$ is a normal subgroup of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$.

Proof. First of all, by Proposition 3.5,

$$\Gamma^+(\mathcal{P})/N \simeq \Gamma^+(\mathcal{P}) \,\square\, \Gamma^+(\mathcal{Q}) \simeq \Gamma^+(\mathcal{Q})/N',$$

so that h really is an isomorphism. Let $f : \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P})$ and $f' : \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{Q})$ be the natural epimorphisms, so that $N' = \ker f$ and $N = \ker f'$ (see Definition 3.4). Then the first part follows directly by Goursat's Lemma [15]. For the last part, note that if $u \in N$ and $v \in N'$, then $uN = N$ and $vN' = N'$. Therefore, $h(uN) = h(N) = N' = vN'$, so that $(u, v) \in \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$. Then we see that $N \times N'$ is a subgroup of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$, and since N is a normal subgroup of $\Gamma^+(\mathcal{P})$ and N' is a normal subgroup of $\Gamma^+(\mathcal{Q})$, it immediately follows that $N \times N'$ (i.e., $X(\mathcal{P}|\mathcal{Q}) \times X(\mathcal{Q}|\mathcal{P})$) is normal in $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$. □

Now we are able to refine Proposition 3.6 to include infinite groups.

Proposition 3.8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes. If $\Gamma^+(\mathcal{P}) \,\square\, \Gamma^+(\mathcal{Q})$ is finite of order k, then the index of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ in $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$ is k.*

Proof. If $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{Q})$ are finite, this follows immediately from the third equation of Proposition 3.6. Now let $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{Q})$ be of arbitrary size, and set $N = X(\mathcal{P}|\mathcal{Q})$ and $N' = X(\mathcal{Q}|\mathcal{P})$, as in Proposition 3.7. If $\Gamma^+(\mathcal{P}) \,\square\, \Gamma^+(\mathcal{Q})$ is finite of order k, then N' has index k in $\Gamma^+(\mathcal{Q})$. For each fixed $u \in \Gamma^+(\mathcal{P})$, Proposition 3.7 says that the element (u, v) is in $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ if and only if $h(uN) = vN'$. In other words, having fixed u we can pick any v that lies in the same (corresponding) coset. Then since N' has index k in $\Gamma^+(\mathcal{Q})$, the set of $(u, v) \in \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ for a fixed u has "index" k in $\{u\} \times \Gamma^+(\mathcal{Q})$. Therefore, letting u range over all elements of $\Gamma^+(\mathcal{P})$, we see that $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ has index k in $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$. □

Corollary 3.9. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes. If $\Gamma^+(\mathcal{P}) \,\square\, \Gamma^+(\mathcal{Q})$ is trivial, then $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.*

3.3 Polytopality of the mix

Our main goal is to use the mixing operation to construct new polytopes. In some cases, we can mix a polytope with a pre-polytope and still get a polytope:

Proposition 3.10. *Let $\mathcal{P}$ be a chiral or directly regular n-polytope with facets isomorphic to $\mathcal{K}$. Let $\mathcal{Q}$ be a chiral or directly regular n-pre-polytope with facets isomorphic to $\mathcal{K}'$. If $\mathcal{K}$ covers $\mathcal{K}'$, then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal.*

Proof. Since $\mathcal{K}$ covers $\mathcal{K}'$, the facets of $\mathcal{P} \diamond \mathcal{Q}$ are isomorphic to $\mathcal{K}$. Therefore, the canonical projection from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P})$ is one-to-one on the subgroup of the facets, and by [2, Lemma 3.2], the group $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ has the intersection property. Therefore, $\mathcal{P} \diamond \mathcal{Q}$ is a polytope. □

In general, there is no guarantee that the mix of a two polytopes is a polytope. For example, for $n \geq 4$, the mix of the n-cube with the n-orthotope is not a polytope [8]. In rank 3, however, polytopality is automatic [7]:

Proposition 3.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular polyhedra (3-polytopes). Then $\mathcal{P} \diamond \mathcal{Q}$ is a chiral or directly regular polyhedron.*

Theorem 3.7 in [7] is one example of a result that works in any rank:

Proposition 3.12. *Let $\mathcal{P}$ be a chiral or directly regular n-polytope of type $\{p_1, \ldots, p_{n-1}\}$, and let $\mathcal{Q}$ be a chiral or directly regular n-polytope of type $\{q_1, \ldots, q_{n-1}\}$. If p_i and q_i are relatively prime for each $i = 2, \ldots, n-2$ (but not neccessarily for $i = 1$ or $i = n-1$), then $\mathcal{P} \diamond \mathcal{Q}$ is a chiral or directly regular n-polytope. Furthermore, if $n \geq 4$, then $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ is a subgroup of index 4 or less in $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.*

We conclude this section with a negative result.

Proposition 3.13. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes. Suppose $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$, and that $\mathcal{Q}$ is of type $\{q_1, \ldots, q_{n-1}\}$. Let $r_i = \gcd(p_i, q_i)$ for $i \in \{1, \ldots, n-1\}$. If there is an integer $m \in \{2, \ldots, n-2\}$ such that $r_{m-1} = r_{m+1} = 1$ and $r_m \geq 3$, then $\mathcal{P} \diamond \mathcal{Q}$ is not a polytope.*

Proof. Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, $\Gamma^+(\mathcal{Q}) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$, and $\beta_i = (\sigma_i, \sigma'_i)$ for each $i \in \{1, \ldots, n-1\}$. To show that $\mathcal{P} \diamond \mathcal{Q}$ is not polytopal, it suffices to show that

$$\langle \beta_{m-1}, \beta_m \rangle \cap \langle \beta_m, \beta_{m+1} \rangle \neq \langle \beta_m \rangle.$$

Now, since p_{m-1} and q_{m-1} are relatively prime, there is an integer k such that $kp_{m-1} \equiv 1 \pmod{q_{m-1}}$. Then since the order of σ_{m-1} is p_{m-1} and the order of σ'_{m-1} is q_{m-1}, we see that

$$\beta_{m-1}^{kp_{m-1}} = (\sigma_{m-1}^{kp_{m-1}}, (\sigma'_{m-1})^{kp_{m-1}}) = (\epsilon, \sigma'_{m-1}),$$

and therefore

$$(\beta_{m-1}^{kp_{m-1}} \beta_m)^2 = (\sigma_m^2, (\sigma'_{m-1}\sigma'_m)^2) = (\sigma_m^2, \epsilon),$$

since we have $(\sigma'_i \sigma'_{i+1})^2 = \epsilon$ for any $i \in \{1, \ldots, n-2\}$. Thus, $(\sigma_m^2, \epsilon) \in \langle \beta_{m-1}, \beta_m \rangle$. Similarly, there is an integer k' such that $k'p_{m+1} \equiv 1 \pmod{q_{m+1}}$, and thus

$$(\beta_m \beta_{m+1}^{k'p_{m+1}})^2 = (\sigma_m^2, (\sigma'_m \sigma'_{m+1})^2) = (\sigma_m^2, \epsilon).$$

Therefore, $(\sigma_m^2, \epsilon) \in \langle \beta_m, \beta_{m+1} \rangle$ as well. So we see that

$$(\sigma_m^2, \epsilon) \in \langle \beta_{m-1}, \beta_m \rangle \cap \langle \beta_m, \beta_{m+1} \rangle.$$

Now, since $r_m \geq 3$, there is no integer k such that $\sigma_m^k = \sigma_m^2$ and $(\sigma'_m)^k = \epsilon$. Therefore, $(\sigma_m^2, \epsilon) \notin \langle \beta_m \rangle$, and that proves the claim. □

4 Measuring invariance

In this section, we develop the theory of internal and external invariance of polytopes; the distinction is similar to that between inner and outer automorphisms of a group. Our framework provides a unified way to measure the extent to which a polytope is chiral, self-dual, or self-Petrie. Our goal is then to understand how the variance of $\mathcal{P} \diamond \mathcal{Q}$ depends on $\mathcal{P}$ and $\mathcal{Q}$, and to use this knowledge to build polytopes with or without specified symmetries.

4.1 External and internal invariance

Our study of invariance starts with the symmetries of

$$W_n := [\infty, \ldots, \infty] = \langle \rho_0, \ldots, \rho_{n-1} \mid \rho_0^2 = \cdots = \rho_{n-1}^2 = \epsilon, (\rho_i \rho_j)^2 = \epsilon \text{ when } |i-j| \geq 2 \rangle,$$

the automorphism group of the universal n-polytope. Let $\mathcal{P}$ be a regular n-polytope with base flag Φ. The group W_n acts on the flags of $\mathcal{P}$ by $\Phi^{j_1,\ldots,j_k}\rho_i = \Phi^{i,j_1,\ldots,j_k}$. If M is the stabilizer of the base flag Φ under this action, then M is normal in W_n and $\Gamma(\mathcal{P}) = W_n/M$.

Suppose φ is in $\mathrm{Aut}(W_n)$, the group of group automorphisms of W_n, and define $\mathcal{P}^\varphi$ to be the flagged poset built from $W_n/\varphi(M)$. If φ fixes M (globally), then $\mathcal{P}$ and $\mathcal{P}^\varphi$ are naturally isomorphic, and we shall consider them equal. On the other hand, if $\varphi(M) \neq M$, then the polytopes $\mathcal{P}$ and $\mathcal{P}^\varphi$ are distinct, and they need not be isomorphic, even though φ induces an isomorphism of their automorphism groups.

Similarly, if $\mathcal{P}$ is a chiral or directly regular n-polytope with base flag Φ, then

$$W_n^+ := [\infty, \ldots, \infty]^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \mid (\sigma_i \cdots \sigma_j)^2 = \epsilon \text{ for } 1 \leq i < j \leq n-1 \rangle$$

acts on the flags of $\mathcal{P}$ by $\Phi^{j_1,\ldots,j_k}\sigma_i = \Phi^{i,i-1,j_1,\ldots,j_k}$. If M is the stabilizer of the base flag under this action, then M is normal in W_n^+ and $\Gamma^+(\mathcal{P}) = W_n^+/M$. Now, taking $\varphi \in \mathrm{Aut}(W_n^+)$, we similarly define $\mathcal{P}^\varphi$ to be the flagged poset built from $W_n^+/\varphi(M)$.

Definition 4.1. *Let $\mathcal{P}$ be a regular or chiral n-polytope (or, more generally, a regular or chiral n-pre-polytope). Let φ be a group automorphism of W_n or W_n^+ (whichever is appropriate), and let $\mathcal{P}^\varphi$ be defined as above.*

(a) *If $\mathcal{P} = \mathcal{P}^\varphi$, we say that $\mathcal{P}$ is* internally φ-invariant*; otherwise we say that $\mathcal{P}$ is* internally φ-variant.

(b) *If $\mathcal{P} \simeq \mathcal{P}^\varphi$, we say that $\mathcal{P}$ is* externally φ-invariant*; otherwise we say that $\mathcal{P}$ is* externally φ-variant.

Of course, if a polytope is internally φ-invariant, it must also be externally φ-invariant. Similarly, if a polytope is externally φ-variant, it must also be internally φ-variant.

Let us consider several applications. Let $\mathcal{P}$ be a regular polytope with $\Gamma(\mathcal{P}) = W_n/M$, and let $\chi_w \in \mathrm{Aut}(W_n)$ be conjugation by $w \in W_n$. Then since M is normal in W_n, χ_w fixes M. Therefore, every regular polytope is internally χ_w-invariant.

Similarly, for any element $w \in W_n$ there is an automorphism $\chi_w \in \mathrm{Aut}(W_n^+)$. If w is even (i.e., if $w \in W_n^+$), then every chiral polytope is internally χ_w-invariant. On the other hand, consider the automorphism $\chi := \chi_{\rho_0}$. This automorphism sends σ_1 to σ_1^{-1} and σ_2 to $\sigma_1^2\sigma_2$ while fixing all other generators σ_i. Then if $\mathcal{P}$ is a chiral polytope, $\mathcal{P}^\chi$ is the enantiomorphic form $\overline{\mathcal{P}}$ of $\mathcal{P}$. In particular, a chiral or directly regular polytope $\mathcal{P}$ is chiral if and only if it is not internally χ-invariant. (But note that in any case, if $\mathcal{P}$ is chiral or directly regular, it is *externally* χ-invariant.)

Moving on, let δ be the automorphism of W_n that sends each ρ_i to ρ_{n-i-1}, and let $\mathcal{P}$ be a regular n-polytope. Then $\mathcal{P}^\delta$ is the dual of $\mathcal{P}$ (and indeed, our notation for the dual was chosen in anticipation of this fact). The polytope $\mathcal{P}$ is externally δ-invariant if and only if it is self-dual. Every regular self-dual polytope has a duality that fixes the base flag while reversing the order [16], and therefore if $\mathcal{P}$ is regular and self-dual, the polytopes $\mathcal{P}$ and $\mathcal{P}^\delta$ have the same flag-stabilizer in W_n. Thus we see that a regular self-dual polytope is always internally δ-invariant.

Similarly, there is an automorphism δ^+ of W_n^+ that sends each σ_i to σ_{n-i}^{-1}. This is the automorphism induced by δ in the previous example (and by an abuse of notation, we frequently use δ to denote this automorphism of W_n^+ as well). Then a directly regular or chiral polytope $\mathcal{P}$ is externally δ^+-invariant if and only if it is self-dual. If $\mathcal{P}$ is properly self-dual (i.e., if there is a duality that fixes the base flag), then it is internally δ^+-invariant; otherwise $\mathcal{P}$ is improperly self-dual and internally δ^+-variant.

For our final example, let π be the automorphism of W_3 that sends ρ_0 to $\rho_0\rho_2$ and fixes every other ρ_i. If $\mathcal{P}$ is a regular polyhedron, then $\mathcal{P}^\pi$ is the Petrie dual of $\mathcal{P}$.

We will now explore the connection between invariance and polytope covers.

Proposition 4.2. *Let $\mathcal{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+, as appropriate. Suppose $\mathcal{Q}$ is a chiral or regular internally φ-invariant n-polytope that covers $\mathcal{P}$. Then $\mathcal{Q}$ covers $\mathcal{P}^\varphi$.*

Proof. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are both regular; the proof is essentially the same in the other cases. We have $\Gamma(\mathcal{P}) = W_n/M$ and $\Gamma(\mathcal{Q}) = W_n/K$ for some normal subgroups M and K of W_n. Since $\mathcal{Q}$ covers $\mathcal{P}$, $K \le M$. Then $\varphi(K) \le \varphi(M)$ as well, and since $\mathcal{Q}$ is internally φ-invariant, $\varphi(K) = K$. Therefore $K \le \varphi(M)$, and so $\mathcal{Q}$ covers $\mathcal{P}^\varphi$. □

Corollary 4.3. *Let $\mathcal{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+, as appropriate. Suppose that φ has finite order k, and that $\mathcal{Q}$ is a chiral or regular internally φ-invariant n-polytope that covers $\mathcal{P}$. Then $\mathcal{Q}$ covers $\mathcal{P} \diamond \mathcal{P}^\varphi \diamond \cdots \diamond \mathcal{P}^{\varphi^{k-1}}$.*

Proof. Repeated application of Proposition 4.2 shows that $\mathcal{Q}$ covers $\mathcal{P}^\varphi$, $\mathcal{P}^{\varphi^2}, \ldots$, and $\mathcal{P}^{\varphi^{k-1}}$. Therefore, by Corollary 3.2, it covers their mix. □

As we shall see shortly, the mix $\mathcal{P} \diamond \mathcal{P}^\varphi \diamond \cdots \diamond \mathcal{P}^{\varphi^{k-1}}$ is actually the minimal internally φ-invariant cover of $\mathcal{P}$. As such, we make the following definition.

Definition 4.4. *Let $\mathcal{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+ (as appropriate) of finite order k. Then we define $\mathcal{P}^{\diamond\varphi}$ to be $\mathcal{P} \diamond \mathcal{P}^\varphi \diamond \cdots \diamond \mathcal{P}^{\varphi^{k-1}}$.*

Proposition 4.5. *Let $\mathcal{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+ (as appropriate) of order finite k. Then $\mathcal{P}^{\diamond\varphi}$ is the minimal chiral or regular internally φ-invariant cover of $\mathcal{P}$.*

Proof. Since $\mathcal{P}^{\varphi^k} = \mathcal{P}$, it is clear that $(\mathcal{P}^{\diamond\varphi})^{\varphi} = \mathcal{P}^{\diamond\varphi}$. So $\mathcal{P}^{\diamond\varphi}$ is internally φ-invariant. By Corollary 4.3, every internally φ-invariant cover of $\mathcal{P}$ must cover $\mathcal{P}^{\diamond\varphi}$ as well. Thus it follows that $\mathcal{P}^{\diamond\varphi}$ is minimal. □

In the rest of Section 4, we will usually assume that φ is an automorphism of W_n^+, and that any polytopes we deal with are chiral or directly regular. Note, however, that the definitions below all still make sense if we work with automorphisms of W_n instead and assume that our polytopes are regular.

Given an automorphism φ of W_n^+ and a chiral or directly regular polytope $\mathcal{P}$, we can consider the variance groups $X(\mathcal{P}|\mathcal{P}^{\varphi})$ and $X(\mathcal{P}^{\varphi}|\mathcal{P})$. By Proposition 3.5, if $\Gamma^+(\mathcal{P}) = W_n^+/M$, then the former is isomorphic to $M\varphi(M)/M$, and the latter is isomorphic to $M\varphi(M)/\varphi(M)$. Since $M \simeq \varphi(M)$, the groups $X(\mathcal{P}|\mathcal{P}^{\varphi})$ and $X(\mathcal{P}^{\varphi}|\mathcal{P})$ are isomorphic. We make the following definition:

Definition 4.6. *Let $\mathcal{P}$ be a chiral or directly regular polytope of rank n. Let $\varphi \in Aut(W_n^+)$. We define*

$$X_{\varphi}(\mathcal{P}) := X(\mathcal{P}|\mathcal{P}^{\varphi}),$$

and we call this the internal φ-variance group of $\mathcal{P}$ *or simply the* φ-variance group of $\mathcal{P}$*.*

In other words, $X_{\varphi}(\mathcal{P})$ is the kernel of the natural epimorphism from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^{\varphi})$ to $\Gamma^+(\mathcal{P}^{\varphi})$ (and also the kernel of the natural epimorphism from $\Gamma^+(\mathcal{P})$ to $\Gamma^+(\mathcal{P}) \,\square\, \Gamma^+(\mathcal{P}^{\varphi})$).

The group $X_{\varphi}(\mathcal{P})$ gives us a measure of how different $\mathcal{P}$ is from $\mathcal{P}^{\varphi}$. At one extreme, $X_{\varphi}(\mathcal{P})$ might be trivial, in which case $\mathcal{P}$ is internally φ-invariant. At the other extreme, $X_{\varphi}(\mathcal{P})$ might coincide with the whole group $\Gamma^+(\mathcal{P})$; in that case, we say that $\mathcal{P}$ is *totally (internally) φ-variant*. (Again, we usually drop the word 'internally' for brevity.)

Let $\mathcal{P}$ be a chiral polytope and let χ be the automorphism of W_n^+ that sends σ_1 to σ_1^{-1} and σ_2 to σ_1^2 while fixing every other σ_i. Then the variance group $X_{\chi}(\mathcal{P})$ is identical to the *chirality group* $X(\mathcal{P})$, introduced in [2] for polytopes and earlier in [1] for maps and hypermaps. We can thus view φ-variance groups as a natural generalization of chirality groups.

4.2 Variance of the mix

Using the tools we have developed, our goal now is to determine how $X_{\varphi}(\mathcal{P} \diamond \mathcal{Q})$ depends on $X_{\varphi}(\mathcal{P})$ and $X_{\varphi}(\mathcal{Q})$. We start with a simple result.

Proposition 4.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $(\mathcal{P} \diamond \mathcal{Q})^{\varphi} = \mathcal{P}^{\varphi} \diamond \mathcal{Q}^{\varphi}$ and $(\mathcal{P} \diamond \mathcal{Q})^{\diamond\varphi} = \mathcal{P}^{\diamond\varphi} \diamond \mathcal{Q}^{\diamond\varphi}$.*

The following lemma completely characterizes the invariance of $\mathcal{P} \diamond \mathcal{Q}$ in terms of polytope covers.

Lemma 4.8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant if and only if it covers $\mathcal{P}^{\diamond\varphi}$ and $\mathcal{Q}^{\diamond\varphi}$.*

Proof. If $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant, then by Corollary 4.3, it covers $(\mathcal{P} \diamond \mathcal{Q})^{\diamond\varphi}$. Furthermore, by Proposition 4.7, the latter polytope is equal to $\mathcal{P}^{\diamond\varphi} \diamond \mathcal{Q}^{\diamond\varphi}$, which covers both $\mathcal{P}^{\diamond\varphi}$ and $\mathcal{Q}^{\diamond\varphi}$. Conversely, if $\mathcal{P} \diamond \mathcal{Q}$ covers $\mathcal{P}^{\diamond\varphi}$ and $\mathcal{Q}^{\diamond\varphi}$, then it covers $(\mathcal{P} \diamond \mathcal{Q})^{\diamond\varphi}$, which itself covers $\mathcal{P} \diamond \mathcal{Q}$. Then we must have that $(\mathcal{P} \diamond \mathcal{Q})^{\diamond\varphi} = \mathcal{P} \diamond \mathcal{Q}$; that is, $\mathcal{P} \diamond \mathcal{Q}$ must be internally φ-invariant. □

Lemma 4.8 has several applications. For example, it tells us that $\mathcal{P} \diamond \mathcal{Q}$ is directly regular if and only if it covers $\mathcal{P} \diamond \overline{\mathcal{P}}$ and $\mathcal{Q} \diamond \overline{\mathcal{Q}}$. Similarly, $\mathcal{P} \diamond \mathcal{Q}$ is properly self-dual if and only if it covers $\mathcal{P} \diamond \mathcal{P}^{\delta}$ and $\mathcal{Q} \diamond \mathcal{Q}^{\delta}$, and it is self-Petrie if and only if it covers $\mathcal{P} \diamond \mathcal{P}^{\pi}$ and $\mathcal{Q} \diamond \mathcal{Q}^{\pi}$.

We now give the main theorem of this section.

Theorem 4.9. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant. Then there is a natural epimorphism from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ to $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^{\varphi})$, and it restricts to an epimorphism from $X(\mathcal{Q}|\mathcal{P})$ to $X_{\varphi}(\mathcal{P}^{\varphi}) = X(\mathcal{P}^{\varphi}|\mathcal{P})$. Similarly, there is a natural epimorphism from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ to $\Gamma^+(\mathcal{Q}) \diamond \Gamma^+(\mathcal{Q}^{\varphi})$ that restricts to an epimorphism from $X(\mathcal{P}|\mathcal{Q})$ to $X_{\varphi}(\mathcal{Q}^{\varphi}) = X(\mathcal{Q}^{\varphi}|\mathcal{Q})$.*

Proof. Let $\Gamma^+(\mathcal{P}) = W_n^+/M$ and $\Gamma^+(\mathcal{Q}) = W_n^+/K$. By Lemma 4.8, since $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant, it covers $\mathcal{P}^{\diamond\varphi}$, which covers $\mathcal{P} \diamond \mathcal{P}^{\varphi}$. Therefore, $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ naturally covers $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^{\varphi})$. Since

$$\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = W_n^+/(M \cap K)$$

and

$$\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^{\varphi}) = W_n^+/(M \cap \varphi(M)),$$

this means that $M \cap K \leq M \cap \varphi(M)$. Thus, the group $M/(M \cap K)$ naturally covers $M/(M \cap \varphi(M))$. By Proposition 3.5, the former is the subgroup $X(\mathcal{Q}|\mathcal{P})$ of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$, and the latter is the subgroup $X(\mathcal{P}^{\varphi}|\mathcal{P})$ of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^{\varphi})$. The result then follows by symmetry. □

Corollary 4.10. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. If $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant, then $|X_{\varphi}(\mathcal{P})|$ divides $|X(\mathcal{Q}|\mathcal{P})|$ and $|X_{\varphi}(\mathcal{Q})|$ divides $|X(\mathcal{P}|\mathcal{Q})|$.*

Corollary 4.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose that $\mathcal{P}$ has infinite φ-variance group $X_{\varphi}(\mathcal{P})$ and that $\mathcal{Q}$ is finite. Then $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-variant (that is, not internally φ-invariant).*

Proof. Since $X(\mathcal{Q}|\mathcal{P})$ is isomorphic to a subgroup of $\Gamma^+(\mathcal{Q})$, it must be finite. Then there is no epimorphism from the finite group $X(\mathcal{Q}|\mathcal{P})$ to the infinite group $X_{\varphi}(\mathcal{P}^{\varphi}) \simeq X_{\varphi}(\mathcal{P})$, and thus by Theorem 4.9, $\mathcal{P} \diamond \mathcal{Q}$ must be internally φ-variant. □

Corollary 4.12. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose $\mathcal{P}$ is internally φ-variant and that $\mathcal{Q}$ has a rotation group $\Gamma^+(\mathcal{Q})$ that is simple. If $X_\varphi(\mathcal{P})$ is not isomorphic to $\Gamma^+(\mathcal{Q})$, then $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-variant.*

Proof. Theorem 4.9 says that if $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant, then $X(\mathcal{Q}|\mathcal{P})$ covers $X_\varphi(\mathcal{P}^\varphi)$. Now, since $\mathcal{P}$ is internally φ-variant, $X_\varphi(\mathcal{P}^\varphi)$ is nontrivial, and since $\Gamma^+(\mathcal{Q})$ is simple, the normal subgroup $X(\mathcal{Q}|\mathcal{P})$ of $\Gamma^+(\mathcal{Q})$ is either trivial or the whole group $\Gamma^+(\mathcal{Q})$. The only way for $X(\mathcal{Q}|\mathcal{P})$ to cover $X_\varphi(\mathcal{P}^\varphi)$ is for $X(\mathcal{Q}|\mathcal{P})$ to be $\Gamma^+(\mathcal{Q})$, and then the only nontrivial group it covers is itself. Therefore, if $X_\varphi(\mathcal{P})$ (and thus $X_\varphi(\mathcal{P}^\varphi)$) is not isomorphic to $\Gamma^+(\mathcal{Q})$, then $X(\mathcal{Q}|\mathcal{P})$ cannot cover $X_\varphi(\mathcal{P}^\varphi)$, and the mix $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-variant. □

We see that there are several simple tests that we can apply to determine whether $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant. We would like to extend the results to the mix of three or more polytopes. In order to do that, however, we need to know more about the size of $X_\varphi(\mathcal{P} \diamond \mathcal{Q})$. The following results are an easy generalization of Lemma 5.5 and Remark 5.1 in [2].

Proposition 4.13. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $X_\varphi(\mathcal{P} \diamond \mathcal{Q})$ is isomorphic to a subgroup of $X_\varphi(\mathcal{P}) \times X_\varphi(\mathcal{Q})$.*

Proposition 4.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose that $\mathcal{Q}$ is internally φ-invariant. Then $X_\varphi(\mathcal{P} \diamond \mathcal{Q})$ is a normal subgroup of $X_\varphi(\mathcal{P})$.*

Next we generalize Corollary 4.10 to find a lower bound for $|X_\varphi(\mathcal{P} \diamond \mathcal{Q})|$.

Theorem 4.15. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $|X_\varphi(\mathcal{P} \diamond \mathcal{Q})|$ is an integer multiple of $|X_\varphi(\mathcal{P})|/|X(\mathcal{Q}|\mathcal{P})|$.*

Proof. Since

$$\mathcal{P} \diamond \mathcal{Q} \diamond \mathcal{P}^\varphi \diamond \mathcal{Q}^\varphi = (\mathcal{P} \diamond \mathcal{Q}) \diamond (\mathcal{P} \diamond \mathcal{Q})^\varphi,$$

the latter covers $\mathcal{P} \diamond \mathcal{P}^\varphi$. Therefore, $|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\varphi)|$ divides $|\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) \diamond \Gamma^+((\mathcal{P} \diamond \mathcal{Q})^\varphi)|$. By Proposition 3.6, the former has size $|\Gamma^+(\mathcal{P})| \cdot |X_\varphi(\mathcal{P})|$, while the latter has size

$$|\Gamma^+(\mathcal{P} \diamond \mathcal{Q})| \cdot |X_\varphi(\mathcal{P} \diamond \mathcal{Q})| = |\Gamma^+(\mathcal{P})| \cdot |X(\mathcal{Q}|\mathcal{P})| \cdot |X_\varphi(\mathcal{P} \diamond \mathcal{Q})|.$$

Therefore, $|X_\varphi(\mathcal{P})|$ divides $|X(\mathcal{Q}|\mathcal{P})| \cdot |X_\varphi(\mathcal{P} \diamond \mathcal{Q})|$, and thus $|X_\varphi(\mathcal{P} \diamond \mathcal{Q})|$ is an integer multiple of $|X_\varphi(\mathcal{P})|/|X(\mathcal{Q}|\mathcal{P})|$. □

Thus we see that, for instance, if $\mathcal{P}$ has a large φ-variance group $X_\varphi(\mathcal{P})$ and if $\mathcal{Q}$ is comparatively small (which forces $X(\mathcal{Q}|\mathcal{P})$ to be small), then $X_\varphi(\mathcal{P} \diamond \mathcal{Q})$ is still large.

A careful refinement lets us make a similar statement about infinite φ-variance groups:

Theorem 4.16. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. If $X(\mathcal{Q}|\mathcal{P})$ is finite and $X_\varphi(\mathcal{P})$ is infinite, then $|X_\varphi(\mathcal{P} \diamond \mathcal{Q})|$ is infinite.*

Proof. Consider the commutative diagram below, where the maps are all the natural epimorphisms:

$$\begin{array}{ccc} \Gamma^+(\mathcal{P} \diamond \mathcal{P}^\varphi \diamond \mathcal{Q} \diamond \mathcal{Q}^\varphi) & \xrightarrow{f_1} & \Gamma^+(\mathcal{P} \diamond \mathcal{P}^\varphi) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ \Gamma^+(\mathcal{P} \diamond \mathcal{Q}) & \xrightarrow[g_2]{} & \Gamma^+(\mathcal{P}) \end{array}$$

Then $\ker(g_1 \circ f_1) = \ker(g_2 \circ f_2)$. Since $X_\varphi(\mathcal{P}) = \ker g_1$ is infinite by assumption, it follows that $\ker(g_1 \circ f_1)$ is infinite. Therefore, $\ker(g_2 \circ f_2)$ is infinite, and thus $\ker g_2$ and $\ker f_2$ cannot both be finite. Now, $\ker f_2 = X_\varphi(\mathcal{P} \diamond \mathcal{Q})$ and $\ker g_2 = X(\mathcal{Q}|\mathcal{P})$. Since $X(\mathcal{Q}|\mathcal{P})$ is finite by assumption, it follows that $X_\varphi(\mathcal{P} \diamond \mathcal{Q})$ must be infinite. □

It is sometimes possible to fully determine $X_\varphi(\mathcal{P} \diamond \mathcal{Q})$:

Theorem 4.17. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose that $\mathcal{P}$ is internally φ-variant, that $X_\varphi(\mathcal{P})$ is simple, and that $\mathcal{Q}$ is internally φ-invariant. If $|X_\varphi(\mathcal{P})|$ does not divide $|\Gamma^+(\mathcal{Q})|$, then $X_\varphi(\mathcal{P} \diamond \mathcal{Q}) = X_\varphi(\mathcal{P})$.*

Proof. Since $|X_\varphi(\mathcal{P})|$ does not divide $|\Gamma^+(\mathcal{Q})|$, the mix $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-variant, by Corollary 4.10. Then by Proposition 4.14, $X_\varphi(\mathcal{P} \diamond \mathcal{Q})$ is a nontrivial normal subgroup of the simple group $X_\varphi(\mathcal{P})$. Therefore, $X_\varphi(\mathcal{P} \diamond \mathcal{Q}) = X_\varphi(\mathcal{P})$. □

Now we will consider the interaction between two automorphisms φ and ψ of W_n^+.

Theorem 4.18. *Let $\mathcal{P}$ be a finite chiral or directly regular n-polytope, and let $\varphi, \psi \in Aut(W_n^+)$ have finite order. If $\mathcal{P} \diamond \mathcal{P}^\psi$ is internally φ-invariant, then $|X_\varphi(\mathcal{P})|$ divides $|X_\psi(\mathcal{P})|$.*

Proof. Apply Corollary 4.10 with $\mathcal{Q} = \mathcal{P}^\psi$. □

Corollary 4.19. *Let $\mathcal{P}$ be a finite chiral n-polytope, and suppose that $|X(\mathcal{P})|$ (that is, $|X_\chi(\mathcal{P})|$) does not divide $|X_\delta(\mathcal{P})|$. Then $\mathcal{P} \diamond \mathcal{P}^\delta$ is a chiral pre-polytope.*

For example, let $\mathcal{P} = \{\{4,4\}_{(1,2)}, \{4,4\}_{(4,2)}\}$, a locally toroidal chiral polytope with $|\Gamma^+(\mathcal{P})| = 480$. Then a calculation with GAP [10] shows that $|X(\mathcal{P})| = 60$ and $|X_\delta(\mathcal{P})| = 4$. Therefore, by Corollary 4.19, $\mathcal{P} \diamond \mathcal{P}^\delta$ is a chiral pre-polytope.

Corollary 4.19 is essentially a restatement of [6, Thm. 5.2], and it highlights one of the principal uses of Theorem 4.18; namely, constructing chiral polytopes with certain external symmetries. Similar methods could be used to construct polyhedra $\mathcal{P}$ such that $\mathcal{P} = \mathcal{P}^{\pi\delta}$ but where $\mathcal{P}$ is neither self-dual or self-Petrie; see [14] for some work on constructing such polyhedra.

Finally, we note that the methods explored here could be somewhat more generalized by working with quotients of groups other than W_n and W_n^+. For example, given a polyhedron $\mathcal{P}$ of type $\{p, q\}$, the group $\Gamma(\mathcal{P})$ can be represented as a quotient of the Coxeter group $[p, q]$, or of $[p, \infty]$. These groups provide new automorphisms that W_n lacks, and would be a further source of external symmetries.

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