Tight chiral polyhedra

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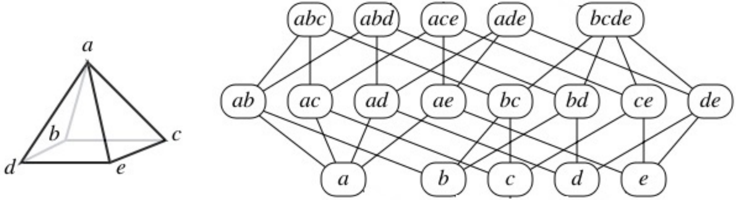
Definition of an abstract polyhedron

An (abstract) polyhedron $\mathcal{P}$ is a ranked poset of vertices (rank 0), edges (rank 1), and faces (rank 2) such that:

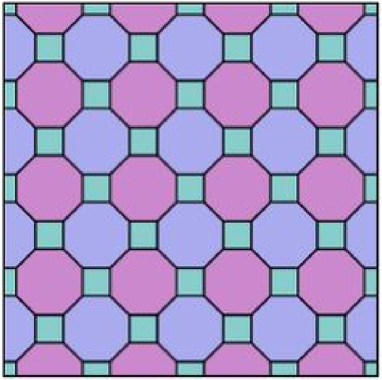
- Every edge is incident to exactly two vertices and two faces.
- Whenever a vertex is incident to a face, there are exactly two edges that are incident to both.
- $\mathcal{P}$ is locally and globally connected.

Examples



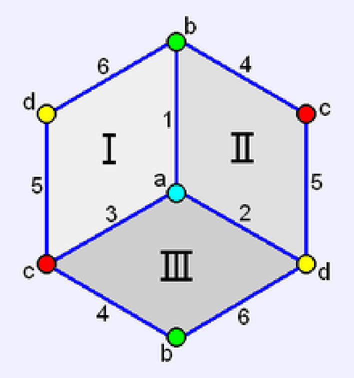
Combinatorial view of a pyramid

Examples



Tiling of the plane by octagons and squares

Examples



The hemicube

Schläfli symbol of a polyhedron

A polyhedron has Schläfli symbol (or type) $\{p, q\}$
if every face is a p-gon and every vertex is q-valent.

Question 1: What is the smallest polyhedron of type $\{p, q\}$?

Size of a polyhedron

A flag of a polyhedron consists of a vertex, edge, and face, all mutually incident.

Proposition

A polyhedron of type $\{p, q\}$ has at least $2pq$ flags.

When a polyhedron of type $\{p, q\}$ has exactly $2pq$ flags, it is called tight.

Tight polyhedra

Theorem (C., 2013)

There is a tight polyhedron of type $\{p, q\}$ if and only if p or q is even.

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Open question!

Tight polyhedra

What is the smallest polyhedron of type $\{p, q\}$ with a prescribed degree of symmetry?

Automorphisms of polyhedra

An automorphism of $\mathcal{P}$ is an order-preserving bijection from $\mathcal{P}$ to itself. The automorphism group of $\mathcal{P}$ is denoted $\Gamma(\mathcal{P})$.

A polyhedron is regular if $\Gamma(\mathcal{P})$ acts transitively on the flags.

Examples: Platonic solids, tiling of the plane by hexagons, hemicube

Tight regular polyhedra

Question 2: For what values of p and q is there a tight regular polyhedron of type $\{p, q\}$?

Tight orientably regular polyhedra

Theorem (Conder and C., 2014)

There is a tight orientably regular polyhedron of type $\{p, q\}$ if and only if one of the following is true:

- *p and q are both even*
- *p is odd and q is an even divisor of $2p$*
- *q is odd and p is an even divisor of $2q$*

Tight non-orientably regular polyhedra

Theorem (C. and Pellicer, 2015)

There is a tight non-orientably regular polyhedron of type $\{p, q\}$ if and only if one of the following is true:

- $p = 4$ *and* $q = 3k$
- $p = 4r$ *and* $q = 6k$, *with* $r > 1$ *odd and* k *odd*
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Chiral polyhedra

A polyhedron $\mathcal{P}$ is chiral if $\Gamma(\mathcal{P})$ has 2 orbits on the flags, and flags that differ in only one element lie in different orbits.

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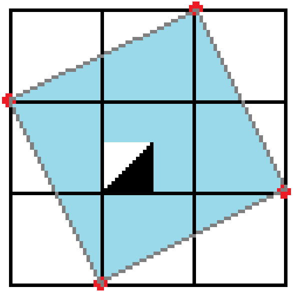
Example:



Automorphism group of chiral polyhedra

Let $\mathcal{P}$ be a chiral polyhedron and pick a base flag Φ.
There is an automorphism σ_1 that rotates the face of Φ one step,
and an automorphism σ_2 that rotates one step around the vertex of Φ.

Furthermore:

- $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$
- It is possible to recover $\mathcal{P}$ from its automorphism group.

Classifying tight chiral polyhedra

Question 3: For what values of p and q is there a tight chiral polyhedron of type $\{p, q\}$?

Searching for tight chiral polyhedra

From Marston Conder's list of chiral polyhedra with up to 2000 flags, we find tight chiral polyhedra of the following types:

$\{6, 9n\}$ for $1 \leq n \leq 18$	$\{8, 32n\}$ for $1 \leq n \leq 3$
$\{9, 18\}$	$\{10, 25n\}$ for $1 \leq n \leq 4$
$\{12, 18n\}$ for $1 \leq n \leq 4$	$\{14, 49\}$
$\{16, 32\}$	$\{18, 6n\}$ for $3 \leq n \leq 4$
$\{18, 9n\}$ for $2 \leq n \leq 6$	$\{20, 50\}$
$\{24, 32\}$	$\{24, 36\}$.

Patterns

Every entry on our table is of one of the following forms:

$$\{2mr, m^2s\} \text{ for odd prime } m$$
$$\{m^2s, 2mr\} \text{ for odd prime } m.$$
$$\{8r, 32s\}$$

Some families of tight chiral polyhedra

Theorem

For each $\beta \geq 2$ *and odd prime* m*, there is a tight chiral polyhedron of type* $\{2m, m^\beta\}$.

Theorem

For each $\beta \geq 5$*, there is a tight chiral polyhedron of type* $\{8, 2^\beta\}$.

Theorem

For each $\beta \geq 5$*, there is a tight chiral polyhedron of type* $\{2^{\beta-1}, 2^\beta\}$.

Polyhedron covers

If $\mathcal{P}$ and $\mathcal{Q}$ are chiral polyhedra, then $\mathcal{P}$ covers $\mathcal{Q}$ if there is a well-defined surjective group homomorphism from $\Gamma(\mathcal{P})$ to $\Gamma(\mathcal{Q})$ sending generators to generators.

(Chiral polyhedra can also cover regular polyhedra via a similar definition.)

Coverings

Proposition

If $\mathcal{P}$ is a tight chiral polyhedron of type $\{p, q\}$ with $q \geq p$, then it covers a tight chiral or regular polyhedron of type $\{p, q'\}$ for some $q' < p$.

We say that the tight chiral polyhedron $\mathcal{P}$ is atomic if it does not cover any other tight chiral polyhedra.

Structure of atomic polyhedra

If H is a subgroup of G, the core of H is the largest subgroup of H that is normal in G. If the core of H is trivial, then H is core-free.

Proposition

Suppose $\mathcal{P}$ is an atomic chiral polyhedron of type $\{p, q\}$ with $q > p$. Then $\langle \sigma_1 \rangle$ is core-free, and $\langle \sigma_2 \rangle$ has a nontrivial core $\langle \sigma_2^{q'} \rangle$ for some q' dividing q.

Structure of atomic polyhedra

Theorem

Suppose $\mathcal{P}$ is an atomic chiral polyhedron of type $\{p, q\}$ with $q > p$, and let $\langle \sigma_2^{q'} \rangle$ be the core of $\langle \sigma_2 \rangle$. Then q/q' is a prime power.

Proof sketch: Suppose $q/q' = bc$ with b and c coprime. Then $\mathcal{P}$ covers tight polyhedra of types $\{p, bq'\}$ and $\{p, cq'\}$. Those must both be regular, because $\mathcal{P}$ is atomic. Then show that this implies that $\mathcal{P}$ is itself regular.

Structure of atomic polyhedra

Lemma

Let $\mathcal{P}$ be an atomic chiral polyhedron of type $\{p, q\}$ with $q > p$, and let $\langle \sigma_2^{q'} \rangle$ be the core of $\langle \sigma_2 \rangle$. Then $\langle \sigma_1^{2q/q'} \rangle$ is normal in $\Gamma(\mathcal{P})$.

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Lemma

Let $\mathcal{P}$ be an atomic chiral polyhedron of type $\{p, q\}$ with $q > p$, and let $\langle \sigma_2^{q'} \rangle$ be the core of $\langle \sigma_2 \rangle$. Then $\langle \sigma_1^{2q/q'} \rangle$ is normal in $\Gamma(\mathcal{P})$.

But $\langle \sigma_1 \rangle$ is core-free, so p divides $2q/q'$.

Since q/q' is a prime power, p is either a power of 2 or twice an odd prime power.

Structure of atomic polyhedra

$\mathcal{P}$ covers a tight regular polyhedron of type $\{p, q'\}$ with $\langle \sigma_2 \rangle$ core-free.

Theorem (C. and Pellicer, 2014)

Suppose $\mathcal{P}$ is a tight orientably regular polyhedron of type $\{p, q'\}$ with $\langle \sigma_2 \rangle$ core-free. Then q' divides p. In particular, for each odd prime dividing p, either q' contains none of the factors of that prime, or it contains all of them.

Structure of atomic polyhedra

Case 1: $q/q' = 2^\beta$.
Then $p = 2^\alpha$, and q' divides p. So q is also a power of 2.

Case 2: $q/q' = m^\beta$ for odd prime m.
Then $p = 2m^\alpha$. q' is either 2 or m^α. It can be shown that $q' \neq 2$. So q is a power of m.

Classification of atomic chiral polyhedra

Theorem

Let $\mathcal{P}$ be an atomic chiral polyhedron of type $\{p, q\}$ with $q > p$. Then the Schläfli symbol of $\mathcal{P}$ is one of the following:

1. $\{2m, m^\beta\}$, *where m is an odd prime and $\beta \geq 2$*
2. $\{8, 2^\beta\}$, *where $\beta \geq 5$*
3. $\{2^{\beta-1}, 2^\beta\}$, *where $\beta \geq 5$.*

Tight chiral polyhedra with q odd

Theorem

Let $\mathcal{P}$ be a tight chiral polyhedron of type $\{p, q\}$ with q odd. Then p is an even divisor of $2q$.

Classification of tight chiral polyhedra

Theorem

There is a tight chiral polyhedron of type $\{p, q\}$ if and only if it has one of the following types or its duals:

- $\{2mr, m^2s\}$, *with s odd and $r | ms$.*
- $\{2mr, m^2s\}$, *with s even.*
- $\{8r, 32s\}$.

Future work

1. Are the tight chiral polyhedra I have found the only ones?
2. What are the Schläfli symbols of tight chiral 4-polytopes?

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Thank you!