Research Statement

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My research interests are in discrete geometry, combinatorial group theory, and graph theory. In particular, I work with abstract polytopes, which are posets that share many key properties with the face-lattices of convex polytopes. Most of my work has focused on either:

1. Developing combinatorial and algebraic tools and techniques for constructing and analyzing abstract polytopes with certain symmetries (see [9, 12, 15, 16, 18]), or
2. Classifying minimal polytopes with respect to certain properties (see [6, 11, 13, 14, 17]).

Computational experimentation using the computer algebra system GAP has also been a key feature of my work, particularly in the analysis of minimal polytopes. I imported the data on automorphism groups of regular and chiral polytopes (see [4, 5]) into GAP, and I wrote several custom functions to extract relevant combinatorial data from the groups.

I have an active research program in abstract polytopes, with several papers in progress and ongoing projects. I would welcome the opportunity to attract students and other faculty to abstract polytopes or related areas. I am also interested in broadening my horizons through colloborations with other mathematicians interested in combinatorics, discrete geometry, and group theory.

1 Abstract polytopes

Here is the basic theory of abstract polytopes, for which [22] is the standard reference. An *(abstract) polytope* $\mathcal{P}$ *of rank* n (also called an *(abstract)* n*-polytope*) is a ranked partially-ordered set that satisfies the following four properties:

1. There is a unique maximal element F_n of rank n and a unique minimal element F_{-1} of rank -1.
2. Each maximal chain contains $n+2$ elements, one in each integer rank from -1 to n.
3. If $F \leq G$ and $\text{rank}(G) - \text{rank}(F) > 2$, then the Hasse diagram of $\{H \mid F < H < G\}$ is a connected graph.

4. (Diamond Condition) If $F \leq G$ and $\text{rank}(G) - \text{rank}(F) = 2$, then there are exactly two elements H such that $F < H < G$.

The elements of a polytope are called *faces*, and a face of rank i is called an *i-face*. In analogy with convex polytopes, we refer to the faces of rank 0, 1, and $n-1$ as *vertices*, *edges*, and *facets*, respectively. The maximal chains of a polytope are called *flags*. If two flags differ in only one face, we say that those flags are *adjacent*, and if that face is an i-face, we say that they are *i-adjacent*. Due to the Diamond Condition, each flag Φ has a unique i-adjacent flag for each i in $\{0, \ldots, n-1\}$, and we denote this flag by Φ^i.

In ranks -1, 0, and 1, there is a unique polytope up to isomorphism (of posets). Each abstract 2-polytope has p vertices and p edges for some p satisfying $2 \leq p \leq \infty$, and there is a unique such polytope for each p. An abstract polyhedron (3-polytopes) can be thought of as a *map*, which is an embedding of a connected (multi)graph into a closed surface such that each connected component of the complement is simply-connected. In general, the face-lattice of any convex polytope or face-to-face tessellation of a space is an abstract polytope, but the theory also includes many new structures with a geometric flavor.

It is also possible to represent an n-polytope $\mathcal{P}$ by its *flag graph*. This is a properly edge-colored n-regular simple graph whose nodes are the flags of $\mathcal{P}$, and where for each flag Φ and each $i \in \{0, \ldots, n-1\}$, there is an edge labeled i from Φ to Φ^i. The poset can be recovered from the flag graph: the i-faces correspond to connected components of the subgraph obtained by deleting all edges with label i, and two faces are related in the poset if the corresponding components have nonempty intersection. Many constructions of new polytopes are more natural to define as operations on a flag graph, rather than on a poset.

If F and G are faces of a polytope such that $F \leq G$, then we define the *section* G/F as

$$G/F = \{H \mid F \leq H \leq G\}.$$

(In the language of posets, G/F is the closed interval $[F, G]$.) The sections of a polytope are themselves polytopes. When we talk about a facet F of an n-polytope, we usually have in mind the section F/F_{-1}, which is an $(n-1)$-polytope. (Recall that F_{-1} is the unique minimal element.) If v is a vertex of an n-polytope, then the *vertex-figure at v* is the section F_n/v, which again is an $(n-1)$-polytope.

An *automorphism* of $\mathcal{P}$ is a bijection that preserves order in both directions (equivalently, a color-preserving graph automorphism of the flag graph), and the automorphism group of $\mathcal{P}$ is denoted by $\Gamma(\mathcal{P})$. There is a natural action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$, and this action is semi-regular (that is, every automorphism is completely determined by where it sends any one flag).

Central to the study of polytopes is the characterization of their symmetries. We say that $\mathcal{P}$ is a *k-orbit polytope* if the action of $\Gamma(\mathcal{P})$ has k orbits on the flags. A polytope is *regular* if it is a 1-orbit polytope (in other words, if the action of $\Gamma(\mathcal{P})$ is transitive on flags). If $\mathcal{P}$ is the face-lattice of a convex polytope, then $\mathcal{P}$ is regular if and only if $\mathcal{P}$ is combinatorially equivalent to a (geometrically) regular convex polytope.

The automorphism group of a regular polytope has a standard form. Given a regular polytope $\mathcal{P}$, we fix a base flag Φ. Then the automorphism group $\Gamma(\mathcal{P})$ is generated by the *abstract reflections* $\rho_0, \dots, \rho_{n-1}$, where ρ_i is the unique automorphism that maps Φ to Φ^i. These generators satisfy $\rho_i^2 = 1$ for all i, and $(\rho_i \rho_j)^2 = 1$ for all i and j such that $|i - j| \geq 2$. Furthermore, the group $\Gamma(\mathcal{P})$ satisfies the following *intersection condition*:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \quad \text{for } I, J \subseteq \{0, \dots, n-1\}. \tag{1}$$

In general, if $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ is a group such that each ρ_i has order 2 and such that $(\rho_i \rho_j)^2 = 1$ whenever $|i-j| \geq 2$, then we say that Γ is a *string group generated by involutions*. If Γ also satisfies the intersection condition (1) given above, then we call Γ a *string C-group*. There is a natural way of building a regular polytope $\mathcal{P}(\Gamma)$ from a string C-group Γ such that $\Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \simeq \mathcal{P}$. In particular, the i-faces of $\mathcal{P}(\Gamma)$ are taken to be the cosets of

$$\Gamma_i := \langle \rho_j \mid j \neq i \rangle,$$

where $\Gamma_i \varphi \leq \Gamma_j \psi$ if and only if $i \leq j$ and $\Gamma_i \varphi \cap \Gamma_j \psi \neq \emptyset$. This construction of a coset geometry is also easily applied to any string group generated by involutions (not just string C-groups), but in that case, the resulting poset is not necessarily a polytope.

2 Chiral polytopes

Among the 2-orbit polytopes, the ones that have received the most attention are the *chiral* polytopes [27]. These are polytopes such that whenever two flags are adjacent, they lie in distinct orbits. Intuitively, this means that chiral polytopes have full rotational symmetry but do not have mirror symmetry. The study of chiral polytopes has its roots in the study of chiral (irreflexible) maps (see [8]) and twisted honeycombs (see [7]). As with regular polytopes, the automorphism groups of chiral polytopes have a standard form, and it is possible to recover the chiral polytope from its automorphism group. Thus, we typically work with chiral polytopes by working with their automorphism groups.

Examples of chiral polytopes have been much more difficult to find than regular polytopes, particularly in ranks 6 and higher. Indeed, until publication of [24], it was unknown whether there were chiral polytopes in every rank. The main difficulty in constructing chiral polytopes highlights an important difference from regular polytopes. Given a regular polytope $\mathcal{K}$, there are infinitely many ways to build a regular polytope $\mathcal{P}$ whose facets are isomorphic to $\mathcal{K}$ (see [23]). On the other hand, if $\mathcal{K}$ is a chiral polytope with chiral facets, then there are no chiral polytopes $\mathcal{P}$ whose facets are isomorphic to $\mathcal{K}$. Thus, it is not possible to repeatedly extend a given chiral polytope to higher and higher ranks – we need genuinely new examples of chiral polytopes in each rank.

In my work, I have described and analyzed several constructions for chiral polytopes. As a graduate student, I built on the work in [1] and [2] to determine when the minimal common cover of two chiral polytopes is itself a chiral polytope [10]. I also used this to construct chiral polytopes that are *self-dual* (isomorphic to the polytope obtained by reversing the partial order) [9]. In [16], Daniel Pellicer and I describe a construction using permutation groups that takes a chiral polytope with regular facets $\mathcal{K}$ and produces a chiral polytope with facets isomorphic to $\mathcal{K}$.

I have also worked on describing the smallest chiral polytopes. In [14], I describe some infinite families of chiral polyhedra that are minimal in some sense (described in the next section), giving presentations and permutation representations for their automorphism groups. Daniel Pellicer and I are currently preparing a manuscript that follows up on this paper to describe the analogous polytopes in rank 4 and to show that there are none in ranks 5 and higher. In [13], I proved that for $n \geq 8$, a chiral n-polytope has at least $48(n-2)(n-2)!$ flags. This lower bound helps explain why the community has had difficulty finding small examples of chiral polytopes in high ranks.

3 Minimal polytopes

One of my main research interests is finding minimal polytopes. I have mainly focused on *equivelar polytopes*, which are defined inductively as follows:

1. Every polygon is equivelar, with *Schläfli symbol* $\{p\}$, where p is how many vertices the polygon has.
2. An n-polytope is equivelar with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ if its facets are all equivelar $(n-1)$-polytopes with Schläfli symbol $\{p_1, \ldots, p_{n-2}\}$ and if its vertex-figures are all equivelar $(n-1)$-polytopes with Schläfli symbol $\{p_2, \ldots, p_{n-1}\}$.

All regular polytopes and chiral polytopes are equivelar, but being equivelar by itself does not imply any degree of symmetry.

In [3], Marston Conder showed that the automorphism group of a regular polytope with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ has order $2p_1 \cdots p_{n-1}$ or greater. In [11], I extended Conder's result to show that any equivelar polytope with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags. A polytope that achieves this lower bound is called *tight*.

In [11], I describe a characterization of tight polytopes using only local information about which faces are incident to each other. I also constructed tight polyhedra of type $\{p, q\}$ for each pair (p, q) with p or q even. The construction amounts to building a q-regular multigraph on p vertices, and designating certain p-cycles as faces, subject to certain restrictions. Further in this direction, I am currently preparing a manuscript which describes a construction using edge-colored graphs that can be used to build tight polytopes in higher ranks, without any particular symmetry requirements.

For more group-theoretic results, Daniel Pellicer and I classified the tight *regular* polyhedra in [17], and Marston Conder and I classified the Schläfli symbols of tight *orientably regular*

n-polytopes [6]. I also classified the Schläfli symbols of tight chiral polyhedra in [14]. These papers rely on the equivalence between regular or chiral polytopes and finitely presented groups in a particular form, and in each case, we look for a small normal subgroup of the automorphism group such that the quotient group is still the automorphism group of a tight polytope. These results contribute to a broader community project to understand the possible structure of the automorphism groups of regular and chiral polytopes.

4 Non-regular polytopes

The early work on abstract polytopes focused on regular and chiral polytopes, in part because the group-theoretic approach was very fruitful. Now there is a growing interest in polytopes that are less symmetric, requiring new approaches. In [15], we defined and examined the *symmetry type graph* of a polytope, a classification tool that has already become standard in the community. In [18], we described a theory of *rooted polytopes* and started the work of adapting the algebraic tools that are commonly used with regular polytopes so that they could be used with non-regular polytopes. Many of these new techniques use either the flag graph of a polytope or the *connection group*, which simultaneously encodes information about the flag graph and about the minimal regular cover of the polytope. I recently submitted an NSF grant proposal with three co-investigators that proposes to further develop these new tools and techniques.

Another problem I am working on with Daniel Pellicer is the classification of finite 3-orbit *skeletal polyhedra* in $\mathbb{E}^2$ and $\mathbb{E}^3$. A skeletal polyhedron is an embedding of a graph into Euclidean space along with a designation of which cycles are faces, subject to some restrictions that ensure that the combinatorial structure is an abstract polyhedron. The regular skeletal polyhedra in $\mathbb{E}^3$ are the Grünbaum-Dress polyhedra (see [19], [20], and [21]), and the chiral skeletal polyhedra in $\mathbb{E}^3$ were classified in [25, 26]. Classifying the 3-orbit skeletal polyhedra is a natural next step, made possible now due to the information about possible symmetry type graphs of 3-orbit polytopes [15].

5 Planned and future research

I have several papers in progress mentioned above, using a variety of approaches. These include:

1. Properties and a partial classification of tight chiral polytopes, using permutation groups and combinatorial group theory,
2. Flat extensions and amalgamations of polytopes, using edge-colored graphs,
3. Polytope operations and regular covers of polytopes, using group theory and graph theory

4. Classification of finite 3-orbit skeletal polyhedra, using discrete geometry.

The first three of these papers I expect to be submitted before next fall.

More broadly, I intend to continue working on the following projects:

1. Adapting algebraic techniques that are used for regular polytopes to combinatorial techniques that could be used regardless of symmetry.

2. Studying and classifying minimal polytopes with certain characteristics.

3. Classifying highly-symmetric skeletal polyhedra in $\mathbb{E}^2$ and $\mathbb{E}^3$.

I am also interested in developing computational tools for working with polytopes and related structures in GAP and Sage. I attended a workshop on Mathematical Data in August (sponsored by the OpenDreamKit project; see `opendreamkit.org`) where I started a collaboration with computer scientists and mathematicians who are developing a central database for catalogs of discrete objects. The goal is to have a robust web interface that allows for filtering data in a variety of ways, and then to have the ability to export the data into a variety of formats so that expert users can perform more analysis in a computer algebra system. I am currently working with two people I met at the workshop to develop a Sage package for maniplexes (a recent generalization of the flag graphs of polytopes).

I like to explore new approaches often. I would be interested in learning more about permutation groups and about algebraic graph theory to see how I can use those results with abstract polytopes. I am also interested in learning more about other structures that are similar to abstract polytopes, such as incidence geometries, and to see how to adapt some of our community's tools to problems in related areas.

Many of the problems I am interested in would be amenable to student research. There are some classification problems where I have a good idea how to get started, and the main tools would be basic group theory, graph theory, and counting principles. These problems are accessible enough that I believe that some "on-the-job" training would be enough to get a student up and running. For a student with a more computational background, I also have some project ideas that would involve generating and analyzing data on certain classes of graphs or finitely presented groups. I would greatly value the opportunity to mentor a student in mathematics research.

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