Research Statement

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Abstract polytopes are combinatorial generalizations of convex polytopes, maps on surfaces, and infinite tessellations. Their study is a rich and varied field, tying together combinatorics, group theory, topology, and geometry. Formally, an abstract polytope $\mathcal{P}$ is a ranked poset that satisfies certain properties that make it similar to the face-lattice of a convex polytope. A polytope is *regular* if the automorphism group (consisting of order-preserving bijections) acts transitively on the maximal chains (which we call *flags*). There is a canonical way to reconstruct a regular polytope from its automorphism group, which makes it possible to study regular polytopes from an entirely group-theoretic point of view. This has led to the development of many interesting techniques for constructing new polytopes.

Much of my work has focused on understanding the covering relations between polytopes. For example, given two or more regular polytopes, what is the minimal regular polytope $\mathcal{P}$ that covers all of them? By working with their automorphism groups, we can find a natural candidate Γ for the automorphism group of the minimal common cover. However, this group sometimes fails to satisfy the properties necessary to be the automorphism group of a polytope. Furthermore, it can be difficult to find a presentation for Γ, which makes it difficult to describe the polytope $\mathcal{P}$. In my dissertation [8], I studied the structure of these minimal common covers in depth, developing techniques to obtain information about $\mathcal{P}$ from Γ and criteria for when Γ is the automorphism group of a regular polytope. Using these results, I described the minimal common covers for every subset of the convex regular polytopes in [10].

One nice application of minimal common covers is the construction of new *chiral polytopes*. Intuitively, a polytope is chiral if it has full symmetry by abstract rotations, but no mirror symmetry. Unlike regular polytopes, chiral polytopes have been difficult to find. One way to produce new examples is to find the minimal common cover of two chiral polytopes. The result is always either chiral or regular, and under certain conditions, we can guarantee that we get a chiral polytope. In [9], I described criteria for when the minimal common cover of two chiral polytopes is chiral, and in [7], I showed how to construct chiral polytopes that are invariant under *duality* (that is, under reversal of the partial order).

In analogy with convex polytopes, every abstract polytope $\mathcal{P}$ of rank n is built out of polytopes of rank $n-1$, which we call the *facets* of $\mathcal{P}$. One of the central questions in the study of chiral polytopes is the *extension problem*: given a regular or chiral polytope $\mathcal{K}$ of rank $n-1$, which chiral polytopes of rank n have facets isomorphic to $\mathcal{K}$? Little progress has been made on this problem, due in part to the fact that if $\mathcal{K}$ itself has chiral facets, then there are no chiral polytopes with facets isomorphic to $\mathcal{K}$. In [13], Daniel Pellicer and I showed that if $\mathcal{K}$ is a chiral polytope with regular facets, then there is always a chiral polytope $\mathcal{P}$ with facets isomorphic to $\mathcal{K}$ such that $\mathcal{P}$ is finite whenever $\mathcal{K}$ is.

An abstract polyhedron (polytope of rank 3) is said to be of *type* $\{p, q\}$ if all of the facets are (face-lattices of) p-gons and all of the vertices have valency q. Each type has a universal polyhedron that covers every other polyhedron of that type. Recently I have turned to considering the smallest polyhedra of each type. In [12], I proved that a polyhedron of type $\{p, q\}$ has at least $2pq$ flags. In the case where either p or q is even (or both), I provided a construction for a polyhedron that meets this lower bound, and showed that this polyhedron is regular if p and q are both even. When p and q are both odd, determining the smallest polyhedron of type $\{p, q\}$ is still open; in this case, it is impossible for a polyhedron to meet the lower bound, because the number of flags of a polyhedron is always divisible by 4. In [4], Marston Conder and I considered the smallest *orientably regular* polyhedra of type $\{p, q\}$, completely determining the values of p and q for which the lower bound is met. In both papers, the results generalize to higher ranks as well.

1 Abstract polytopes

We use several definitions from [17, Ch. 2]. An *(abstract) polytope $\mathcal{P}$ of rank n* (also called an *(abstract) n-polytope*) is a ranked partially-ordered set that satisfies the following four properties:

(1) There is a unique maximal element, which has rank n, and a unique minimal element, which has rank -1.

(2) Each maximal chain contains $n+2$ elements, one in each integer rank from -1 to n.

(3) If $F \leq G$ and $\text{rank}(G) - \text{rank}(F) > 2$, then the Hasse diagram of $\{H \mid F < H < G\}$ is a connected graph.

(4) If $F \leq G$ and $\text{rank}(G) - \text{rank}(F) = 2$, then there are exactly two elements H such that $F < H < G$.

The elements of a polytope are called *faces*, and a face of rank i is called a *i-face*. In analogy with convex polytopes, we refer to the faces of rank 0, 1, and $n-1$ as *vertices*, *edges*, and *facets*, respectively. The maximal chains of a polytope are called *flags*. If two flags differ in only one face, we say that those flags are *adjacent*, and if that face is an i-face, we say that they are *i-adjacent*. Due to the fourth defining property, each flag Φ has a unique i-adjacent flag for each i in $\{0, \ldots, n-1\}$, and we denote this flag by Φ^i.

In ranks -1, 0, and 1, there is a unique polytope up to isomorphism (of posets). Abstract polytopes of rank 2 are called *abstract polygons*. For each $2 \leq p \leq \infty$, there is a unique abstract polygon (up to isomorphism) with p vertices and p edges, denoted $\{p\}$. Furthermore, every abstract polygon is of this type. In general, the face-lattice of any convex polytope or face-to-face tessellation is an abstract polytope, but the theory also includes many new and interesting structures with a geometric flavor.

If F and G are faces of $\mathcal{P}$ such that $F \leq G$, then the *section* G/F consists of those faces H such that $F \leq H \leq G$. If F is an $(i-2)$-face of $\mathcal{P}$ and G is an $(i+1)$-face of $\mathcal{P}$ such that $F < G$, then the section G/F is an abstract polygon. If for each $1 \leq i \leq n-1$, the section G/F is equal to $\{p_i\}$, no matter which $(i-2)$-face F and $(i+1)$-face G we choose, then we say that $\mathcal{P}$ has *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$, or that $\mathcal{P}$ is *of type* $\{p_1, \ldots, p_{n-1}\}$. If $\mathcal{P}$ has a Schläfli symbol, then we say that $\mathcal{P}$ is *equivelar*.

We say that $\mathcal{P}$ *covers* $\mathcal{Q}$ if there is a surjective function $\gamma : \mathcal{P} \to \mathcal{Q}$ that preserves flag adjacency and order and rank of faces. An *automorphism* of $\mathcal{P}$ is an order-preserving bijection, and we denote the automorphism group of $\mathcal{P}$ by $\Gamma(\mathcal{P})$. There is a natural action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$, and every automorphism is completely determined by where it sends any one flag. In particular, the only automorphism that fixes any flags is the identity.

Central to the study of polytopes is the characterization of their symmetries. We say that $\mathcal{P}$ is a *k-orbit polytope* if the action of $\Gamma(\mathcal{P})$ has k orbits on the flags. A polytope is *regular* if it is a 1-orbit polytope (in other words, if the action of $\Gamma(\mathcal{P})$ is transitive). If $\mathcal{P}$ is the face-lattice of a convex polytope, then $\mathcal{P}$ is regular if and only if $\mathcal{P}$ is combinatorially equivalent to a (geometrically) regular convex polytope.

The automorphism group of a regular polytope has a particularly nice form. Given a regular polytope $\mathcal{P}$, we fix a base flag Φ. Then the automorphism group $\Gamma(\mathcal{P})$ is generated by the *abstract reflections* $\rho_0, \ldots, \rho_{n-1}$, where ρ_i is the unique automorphism that maps Φ to Φ^i. These generators satisfy $\rho_i^2 = \varepsilon$ for all i, and $(\rho_i \rho_j)^2 = \varepsilon$ for all i and j such that $|i-j| \geq 2$. Furthermore, the group $\Gamma(\mathcal{P})$ satisfies the following *intersection condition*:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \quad \text{for } I, J \subseteq \{0, \ldots, n-1\}. \tag{1}$$

In general, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a group such that each ρ_i has order 2 and such that $(\rho_i \rho_j)^2 = \varepsilon$ whenever $|i-j| \geq 2$, then we say that Γ is a *string group generated by involutions* (or *sggi*). If Γ also satisfies the intersection condition (1) given above, then we call Γ a *string C-group*. There is a natural way of building a regular polytope $\mathcal{P}(\Gamma)$ from a string C-group Γ such that $\Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \simeq \mathcal{P}$. In particular, the i-faces of $\mathcal{P}(\Gamma)$ are taken to be the cosets of

$$\Gamma_i := \langle \rho_j \mid j \neq i \rangle,$$

where $\Gamma_i \varphi \leq \Gamma_j \psi$ if and only if $i \leq j$ and $\Gamma_i \varphi \cap \Gamma_j \psi \neq \emptyset$. This construction is also easily applied to any sggi (not just string C-groups), but in that case, the resulting poset is not necessarily a polytope.

The *rotation subgroup* of $\Gamma(\mathcal{P})$, denoted $\Gamma^+(\mathcal{P})$, consists of all automorphisms that can be written as words of even length in terms of the generators ρ_i. The index of $\Gamma^+(\mathcal{P})$ in $\Gamma(\mathcal{P})$ is at most 2, and when it is equal to 2, we say that $\mathcal{P}$ is *orientably regular*.

Among the 2-orbit polytopes, the ones that have received the most attention are the *chiral* polytopes [25]. These are polytopes such that whenever two flags are adjacent, they lie in distinct orbits. Intuitively, this means that chiral polytopes have full rotational symmetry but do not have mirror symmetry. The study of chiral polytopes has its roots in the study of chiral (irreflexible) maps (see [6]) and twisted honeycombs (see [5]).

The automorphism group of a chiral n-polytope is generated by *abstract rotations* $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i sends a chosen base flag Φ to $(\Phi^i)^{i-1}$. For any $i < j$, the automorphism $\sigma_i \sigma_{i+1} \cdots \sigma_j$ has order 2. Furthermore, the group $\Gamma(\mathcal{P}) = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle$ satisfies an intersection condition that is analogous to Equation 1 (but somewhat more technical). As with regular polytopes, if $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ is any group that satisfies this intersection condition and such that $\sigma_i \sigma_{i+1} \cdots \sigma_j$ has order 2 for each $i < j$, then there is a natural way to build a polytope directly from the group. In this case, the resulting polytope will either be regular or chiral.

Examples of chiral polytopes have been much more difficult to find than regular polytopes. Though we have many examples in ranks 3 and 4, only a handful of examples are known in ranks 5 and higher. Indeed, until publication of [26], it was unknown whether there were chiral polytopes in every rank. The main difficulty in constructing chiral polytopes highlights an important difference from regular polytopes. Given a regular polytope $\mathcal{K}$, there are infinitely many ways to build a regular polytope $\mathcal{P}$ whose facets are isomorphic to $\mathcal{K}$ (see [21]). On the other hand, if $\mathcal{K}$ is a chiral polytope with chiral facets, then there are no chiral polytopes $\mathcal{P}$ whose facets are isomorphic to $\mathcal{K}$. Thus, we need genuinely new examples of chiral polytopes in each rank.

2 Mixing polytopes

My thesis research focused on a method for finding the minimal common cover of regular or chiral polytopes $\mathcal{P}$ and $\mathcal{Q}$. We do so using the *mix* of two polytopes, first introduced for regular polytopes in [18] and for chiral polytopes in [2]. The idea is to find the minimal group that covers $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ (using the mix or *parallel product* of the groups, see [28]), and to build a polytope from that. The main problem is that the mix of the groups $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ may not satisfy the required intersection condition, and thus may not actually be the automorphism group of a polytope. Another difficulty is determining the structure of the mix, including the size of the automorphism group. My research has handled both difficulties in detail.

The mixing operation proceeds as follows. Given groups $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Lambda = \langle y_1, \ldots, y_n \rangle$, we define the mix of Γ and Λ (denoted $\Gamma \diamond \Lambda$) to be the subgroup of $\Gamma \times \Lambda$ generated by the diagonal elements $z_i := (x_i, y_i)$. It is easy to show that $\Gamma \diamond \Lambda$ is the minimal group that covers Γ and Λ via epimorphisms sending z_i to x_i and to y_i, respectively.

If $\mathcal{P}$ and $\mathcal{Q}$ are regular or chiral n-polytopes, we define the mix of $\mathcal{P}$ and $\mathcal{Q}$ to be the poset built from $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ using the construction in Section 1. If $\mathcal{P}$ and $\mathcal{Q}$ are regular or chiral polyhedra (that is, polytopes of rank 3), then $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$ always satisfies the necessary intersection condition, and so in that case, $\mathcal{P} \diamond \mathcal{Q}$ is always itself a regular or chiral polyhedron. In higher ranks, however, the poset $\mathcal{P} \diamond \mathcal{Q}$ can fail to be a polytope.

We can naturally extend the definition of the mix to any finite number of polytopes. In [10], I describe the mix of every subset of regular convex polytopes (of a given rank), listing all of the combinatorial data and determining when the mix is a polytope. In particular, I show that for $n \geq 4$, the mix of the n-cube with the n-orthotope is not a polytope, and that the mix of the n-simplex with either the n-cube or the n-orthotope (or both) is a polytope.

One particularly useful application of mixing is in the construction of chiral polytopes. If the mix of two chiral polytopes is itself a polytope, then it must be either regular or chiral. Since we have so few examples of chiral polytopes, particularly in high rank, we are interested in determining what conditions ensure that the mix is chiral. In [9], I developed several such criteria, using the *chirality group* of a polytope, defined in [1] for maps and in [2] for polytopes. If $\mathcal{P}$ is a chiral polytope, then its chirality group $X(\mathcal{P})$ is a nontrivial normal subgroup of $\Gamma(\mathcal{P})$ that measures how far away $\mathcal{P}$ is from being regular. I proved in [9] that if $\mathcal{P}$ and $\mathcal{Q}$ are finite chiral polytopes such that $|X(\mathcal{P})|$ does not divide $|\Gamma(\mathcal{Q})|$, then $\mathcal{P} \diamond \mathcal{Q}$ is chiral. In fact, this is just one consequence of the more general result that if $\mathcal{R}$ is the minimal regular cover of $\mathcal{P}$, then $|\Gamma(\mathcal{R})| = 2|\Gamma(\mathcal{P})| \cdot |X(\mathcal{P})|$, and any other regular polytope that covers $\mathcal{P}$ must also cover $\mathcal{R}$.

Mixing is also useful for constructing polytopes with certain "external" symmetries. Each polytope $\mathcal{P}$ has a *dual*, $\mathcal{P}^\delta$, obtained by reversing the partial order. If $\mathcal{P}$ is isomorphic to $\mathcal{P}^\delta$, we say that $\mathcal{P}$ is *self-dual*. It is easy to show that the mix of $\mathcal{P}$ with $\mathcal{P}^\delta$ is self-dual; in fact, it is the minimal self-dual cover of $\mathcal{P}$. In [7], I consider the mix $\mathcal{P} \diamond \mathcal{P}^\delta$ in the case where $\mathcal{P}$ is chiral. I am chiefly interested in the case where the mix is a chiral polytope. Unfortunately, the main result from [9] does not help, since $X(\mathcal{P})$ is isomorphic to $X(\mathcal{P}^\delta)$, and thus it is always the case that $|X(\mathcal{P})|$ divides $|\Gamma(\mathcal{P}^\delta)|$. Nevertheless, there are some similar criteria that work. For example, if $|\Gamma(\mathcal{P} \diamond \mathcal{P}^\delta)| < |\Gamma(\mathcal{P})| \cdot |X(\mathcal{P})|$, then $\mathcal{P} \diamond \mathcal{P}^\delta$ must be chiral. Using this result and others like it, we are able to describe some new self-dual chiral polytopes.

There is another interesting external symmetry that has been studied in polyhedra and maps (see [16, 27]). We define the *Petrie polygons* of a regular polyhedron to be edge-circuits such that every two consecutive edges lie in a common face, but no three consecutive edges do. Then given a polyhedron $\mathcal{P}$, its *Petrie dual* $\mathcal{P}^\pi$ consists of the same vertices and edges as $\mathcal{P}$, but its faces are the Petrie polygons of $\mathcal{P}$. Taking the Petrie dual of a polyhedron also forces the old faces to be the new Petrie polygons, so that $\mathcal{P}^{\pi\pi} \simeq \mathcal{P}$. If $\mathcal{P}$ is isomorphic to $\mathcal{P}^{\pi}$, then we say that $\mathcal{P}$ is *self-Petrie*. The external symmetries of duality and Petrie duality form a group of order 6, and if we mix $\mathcal{P}$ with all of its images under this group, we obtain the minimal self-dual, self-Petrie cover of $\mathcal{P}$. In this case, since the mix of two regular polyhedra is always a regular polyhedron, the only difficulty lies in determining the structure of the resulting polyhedron. Indeed, the problem of describing the mix of 6 polyhedra in any sort of explicit terms (say, by giving a presentation for the group) seems completely intractable in general; the group is simply too large for any meaningful calculations. In certain cases, depending on simple combinatorial data such as the length of the Petrie polygons of $\mathcal{P}$ and the valency of each vertex, it is possible to dodge these complications and describe the automorphism group of the cover. In [11], I present several such conditions, and using these I describe the self-dual, self-Petrie covers of many interesting finite regular polyhedra.

3 Chiral extensions

Chiral polytopes have only recently become an active area of study, and many fundamental questions remain unanswered. Problems 24-30 in [23] all concern the *extension problem* for chiral polytopes, which asks: given a regular or chiral n-polytope $\mathcal{K}$, what chiral $(n+1)$-polytopes $\mathcal{P}$ exist with facets isomorphic to $\mathcal{K}$? Such a polytope $\mathcal{P}$ is called an *extension* of $\mathcal{K}$. If $\mathcal{K}$ is a chiral polytope with chiral facets, then no chiral extension of $\mathcal{K}$ exists, since the $(n-2)$-faces of a chiral n-polytope are always regular. On the other hand, if $\mathcal{K}$ is a chiral polytope with regular facets, then there is always at least one chiral extension, described in [26]. This is the *universal* extension of $\mathcal{K}$, which covers every other extension of $\mathcal{K}$ and is always infinite.

In [13], Daniel Pellicer and I provide a construction for an extension $\mathcal{P}$ of a chiral polytope $\mathcal{K}$ with regular facets. Furthermore, this construction has the property that if $\mathcal{K}$ is finite, then so is $\mathcal{P}$ (and in particular, if $\mathcal{K}$ has N flags, then $\mathcal{P}$ has at most $N!$). The technique involves constructing a permutation group that is then shown to be the automorphism group of a chiral polytope. This is accomplished with the help of *Generalized Permutation Representation (GPR) graphs*, which completely encode the action of a group on a specified set. Our work builds on Daniel's earlier work in [22], where he used GPR graphs to build chiral extensions of regular polytopes $\mathcal{K}$ in the case where $\mathcal{K}$ is the minimal regular cover of a chiral polytope. The idea (in both papers) is to take several copies of a GPR graph for the n-polytope $\mathcal{K}$, and then add some new edges to produce a GPR graph for an $(n+1)$-polytope $\mathcal{P}$. By choosing the new edges carefully, we can ensure that the resulting permutation group is the automorphism group of a chiral polytope.

4 Minimal equivelar polytopes

In every rank, the smallest n-polytope is a regular polytope of type $\{2, 2, \ldots, 2\}$, with 2^n flags. However, when considering polytopes of type $\{p_1, \ldots, p_{n-1}\}$, we often assume that each p_i is at least 3, since whenever some p_i is equal to 2, the automorphism group decomposes into a direct product. So we ask: what is the smallest regular polytope in each rank if no p_i is allowed to be 2? This was the question considered by Marston Conder in [3]. He started by showing that a regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags. Polytopes that meet this lower bound are called *tight*. Conder then exhibited a family of tight polytopes, one in each rank, of type $\{4, \ldots, 4\}$. Using properties of the automorphism groups of regular polytopes, he showed that these were the smallest regular polytopes in rank $n \geq 9$, and that in every other rank, the minimum was also attained by a tight polytope.

In [12], I showed that the bound on the number of flags applied to any equivelar polytope, regardless of regularity. Therefore, it makes sense to extend the definition of a tight polytope to include any equivelar polytope of type $\{p_1, \ldots, p_{n-1}\}$ with $2p_1 \cdots p_{n-1}$ flags. As an alternative characterization, I showed that an equivelar polytope is tight if and only if every i-face is incident with every $(i+2)$-face, for each i in $\{0, \ldots, n-3\}$. This essentially turns the global criterion for tightness into a series of local criteria, which is helpful when trying to construct tight polytopes.

Tightness is a restrictive property, and not every Schläfli symbol has a corresponding tight polytope. In order for there to be a tight polytope of type $\{p_1, \ldots, p_{n-1}\}$, it is necessary that no two adjacent values p_i and p_{i+1} both be odd. Theorem 5.1 in [12] shows that this condition is sufficient in rank 3 by explicitly describing tight polyhedra of type $\{p_1, p_2\}$ when p_1 or p_2 is even. In higher ranks, the question of sufficiency is still open.

A related question is to determine what conditions on p_i are necessary and sufficient for there to be a *regular* tight polytope of type $\{p_1, \ldots, p_{n-1}\}$. In [12], I showed that there is a regular tight polytope of type $\{p_1, \ldots, p_{n-1}\}$ whenever every p_i is even. Using the regular polytope atlas at [14], I also investigated regular tight polyhedra. I found that there are regular tight polyhedra of type $\{3, 4\}$ and $\{3, 6\}$, but none of type $\{3, k\}$ for any k with $7 \leq k \leq 333$. Similar data for other odd values of p_1 led me to conjecture that, if p_1 is odd and $p_2 > 2p_1$, then there is no regular tight polyhedron of type $\{p_1, p_2\}$.

In recent work with Marston Conder, we have confirmed the conjecture for *orientably* regular polyhedra (see [4]). In fact, something stronger is true: if p_1 is odd, then there is an orientably regular tight polyhedron of type $\{p_1, p_2\}$ if and only if p_2 is an even divisor of $2p_1$. Using this as a base, we were able to fully characterize the Schläfli symbols for which there is an orientably regular tight polytope of that type.

5 Future research

I have several research projects planned for the short term. In the area of chiral extensions, Daniel Pellicer and I would like to determine which regular polytopes can be extended to chiral polytopes. Though this work will build on our previous work with GPR graphs, it is somewhat more difficult to build a chiral polytope from a regular one. Essentially, when extending a chiral polytope, you get chirality "for free", whereas when extending a regular polytope, you have to build in the chirality somehow. It is for this reason that Pellicer's earlier paper [22] only applies to a very particular class of regular polytope, but we are hopeful that with further work, the conditions on the regular polytope could be relaxed.

I would also like to continue working with tight polytopes. Some of the unanswered questions in this area are purely combinatorial, such as determining which Schläfli symbols $\{p_1, \ldots, p_{n-1}\}$ appear among tight polytopes (with no restriction on how symmetric the polytope is). Other questions are essentially group-theoretic, such as classifying the nonorientably regular tight polytopes. Studying tight chiral polytopes would also be interesting, and could potentially provide many new examples of small chiral polytopes.

One new area that I am starting to work in is the problem of determining the minimal regular polytopes that cover a given polytope $\mathcal{P}$. The minimal regular covers of the prisms and antiprisms were found in [15], and the minimal regular covers of the Archimedean tilings were found in [24, 19]. More generally, it was recently shown in [20] that every finite polytope has a finite regular cover. I am currently working on bounding the size of the smallest regular cover of $\mathcal{P}$ using only some simple data from $\mathcal{P}$, such as the size of the automorphism group and the number of flag orbits. In certain cases, such as when $\Gamma(\mathcal{P})$ is a simple group, I can determine the structure of the minimal regular cover exactly.

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