Finite 3-orbit polyhedra in ordinary space, II

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Abstract

We enumerate the 188 3-orbit skeletal polyhedra in $\mathbb{E}^3$ with irreducible symmetry group. The analysis is carried out by determining the polyhedra having each irreducible finite group of isometries as their symmetry group. Relevant information of every polyhedron is also organized in tables.

Key Words: Skeletal polyhedron, 3-orbit polyhedron, operations on polyhedra, 3-dimensional finite irreducible symmetry groups.

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1 Introduction

The study of symmetries of polyhedra has a long history. Already the Greeks knew about all Platonic and Archimedean solids that in current terminology are the only vertex-transitive convex polyhedra whose faces are all regular, other than the prisms and antiprisms. Without stating proper definitions, the Greeks understood convexity as a defining condition for polyhedra.

Centuries later, a deeper understanding of the properties of symmetry of polyhedra led to modifications of the notion of 'polyhedron'. First the two stellated dodecahedra and then the great dodecahedron and great icosahedron were recognized as regular polyhedra (see [5]).

The twentieth century brought formal definitions and interactions among distinct areas or mathematics. This is the time of the appearance of the Petrie-Coxeter polyhedra (see [4]), and decades later of Grünbaum's definition of what are known today as skeletal polyhedra (see [13]). Here regularity is formally defined in terms of a group action.

Most of the study of symmetries of skeletal polyhedra has focused on the regular ones. Their full classification was given in the twentieth century in [10] and [11]. Later generalizations to higher dimensions and other geometries were studied (see for example [1], [2], [16], [19]).

Fewer studies have been performed on highly symmetric non-regular polyhedra that are not convex. They include the enumeration of the starry uniform polyhedra with planar faces [6], the classification of chiral polyhedra [24, 25] and the polyhedra obtained by Wythoffian operations from regular ones [26, 28].

One of the natural ways to proceed when studying non-regular polyhedra is to enumerate them according to the number of flag orbits (regular polyhedra have only one orbit). The enumeration of the finite 2-orbit polyhedra was announced [15]. That of the infinite 2-orbit polyhedra is ongoing [21] and is a logical follow-up to the classification of the chiral polyhedra.

In [9] the authors enumerate the finite 3-orbit polyhedra with reducible symmetry group. In this paper we complete the enumeration of finite 3-orbit polyhedra by studying those with irreducible symmetry group.

The paper is organized as follows. Section 2 is about basic concepts and results on skeletal polyhedra. The 7 finite irreducible groups of isometries of $\mathbb{E}^3$ and some of their properties are explained in Section 3. The approach we will follow towards the enumeration of the polyhedra is explained in Section 4, including general theory of those polyhedra that admit a continuous movement that preserves the symmetry group and the combinatorics of the polyhedra, without being homothety or orientation preserving isometries. The enumeration is carried out in Sections 5, 6, 7, 8, 9; each section deals with one symmetry group. We conclude with final remarks in Section 10, and information about the polyhedra is collected in tables in an appendix at the end of the paper.

2 Polygons and polyhedra

Here we recall the main ideas and basic results on polygons, polyhedra and 3-orbit polyhedra. More details can be found in [9].

A *skeletal polygon* is an embedding of a finite, connected 2-regular graph in Euclidean space $\mathbb{E}^3$ that is one-to-one on the vertex set. The images of the edges are line segments, and distinct edges are required to correspond to distinct segments. As a consequence of this, no polygon can have fewer than 3 edges. If two edges have no vertex in common then they may intersect in an interior point. Polygons need not be convex nor planar. Even though in general polygons are allowed to be infinite, here we will only consider finite ones. Henceforth, we assume that every polygon is an embedding of a cycle in $\mathbb{E}^3$.

A *symmetry* of a polygon $\mathcal{Q}$ is an isometry of $\mathbb{E}^3$ preserving $\mathcal{Q}$ setwise. If $\mathcal{Q}$ has n vertices and its symmetry group has $2n$ elements acting on the vertex set as the standard actions of the $2n$ elements of the dihedral group D_n, we say that $\mathcal{Q}$ is a *regular polygon*. Regular polygons can be planar or skew. Planar regular polygons may be convex or star-shaped, whereas skew polygons are obtained from planar polygons by blending them with a line segment in the sense of [18, Chapter 5A]. For more details on regular polygons see [13].

For a given collection $\mathcal{X}$ of skeletal polygons and a given vertex v of some polygon in $\mathcal{X}$, the *vertex-figure at v* is a graph whose vertices are the neighbors of v, two of which are joined by a line segment if and only if they are the neighbors of v in some polygon in $\mathcal{X}$.

A *skeletal polyhedron* $\mathcal{P}$ is a collection of polygons (also called *faces*) satisfying the following properties:

- Every compact subset of $\mathbb{E}^3$ contains finitely many vertices of (the faces of) $\mathcal{P}$.
- The graph induced by the vertices and edges of all polygons is connected.
- The vertex-figure at every vertex is a skeletal polygon.

As a consequence of the last item, every vertex of a skeletal polyhedron has degree at least 3, and every edge of a skeletal polyhedron belongs to precisely two faces. When convenient we may refer to the vertices, edges and faces of $\mathcal{P}$ by 0-faces, 1-faces and 2-faces, respectively. The 1-*skeleton* of $\mathcal{P}$ consists of the sets of vertices and edges of $\mathcal{P}$.

Skeletal polygons and polyhedra correspond respectively to faithful realizations of abstract polygons and abstract polyhedra in $\mathbb{E}^3$ as defined in [18, Chapter 5]. Unless explicitly stated, by 'polygon' and 'polyhedron' we shall understand 'skeletal polygon' and 'skeletal polyhedron', respectively. In this paper, all polyhedra that we consider are finite.

A *flag* of a polyhedron $\mathcal{P}$ is a triple of incident vertex, edge and face. Flags that differ in exactly one element are called *adjacent*, and i-*adjacent* if they differ precisely in the i-face. The axioms of polyhedra imply that for every $i \in \{0, 1, 2\}$ and for every flag Φ there exists a unique i-adjacent flag of Φ, and we shall denote it by Φ^i.

A polyhedron $\mathcal{P}$ is *equivelar* if there exist integers p and q such that all its faces are p-gons and all its vertices are q-valent. In such cases the Schläfli type of $\mathcal{P}$ is defined as the ordered pair $\{p, q\}$.

A *dual* of $\mathcal{P}$ is a polyhedron $\mathcal{P}^\delta$ where for $i \in \{0, 1, 2\}$ there is a bijection between the set of i-faces of $\mathcal{P}$ and the set of $(2-i)$-faces of $\mathcal{P}^\delta$, where an i-face F is contained in a j-face G of $\mathcal{P}$ if and only if the $(2-j)$-face of $\mathcal{P}^\delta$ corresponding to G is contained in the $(2-i)$-face of $\mathcal{P}^\delta$ corresponding to F. Many notions of polyhedra have a natural dual notion that interchanges the role of vertices and faces.

A *Petrie walk* of a polyhedron $\mathcal{P}$ is a closed walk on the 1-skeleton of $\mathcal{P}$ where every two consecutive edges belong to a face, but three consecutive edges never do. The *Petrial* of $\mathcal{P}$ is the collection $\mathcal{P}^\pi$ of Petrie walks of $\mathcal{P}$. Its sets of vertices, edges and vertex-figures are the same as those of $\mathcal{P}$, and so it may fail the definition of polyhedron only when the Petrie walks of $\mathcal{P}$ are not polygons. In case that $\mathcal{P}^\pi$ is a polyhedron then $(\mathcal{P}^\pi)^\pi = \mathcal{P}$.

A *symmetry* of a polyhedron $\mathcal{P}$ is an isometry preserving $\mathcal{P}$. We shall denote the group of symmetries of $\mathcal{P}$ by $G(\mathcal{P})$. If the affine span of $\mathcal{P}$ has dimension 3 (as it will be the case for all polyhedra in this paper) then the flag stabilizers under $G(\mathcal{P})$ must be trivial. If $G(\mathcal{P})$ induces k orbits on flags then $\mathcal{P}$ is said to be a *k-orbit polyhedron*; 1-orbit polyhedra are called *regular*.

The *symmetry type graph* $T(\mathcal{P})$ of $\mathcal{P}$ is a connected pre-graph (allowing semi-edges and multiple edges) with edges labeled in $\{0, 1, 2\}$. The vertex set of $T(\mathcal{P})$ is the set of flags-orbits of $\mathcal{P}$. Two vertices X_1 and X_2 of $T(\mathcal{P})$ are joined by an edge labeled i whenever a flag in X_1 is i-adjacent to a flag in X_2. In addition, if two flags in the same orbit are i-adjacent then there is a semi-edge labeled i at the corresponding vertex.

Symmetry type graphs illustrate the local configuration of flags according to their flag orbits. The connected components of $T(\mathcal{P})$ after removing the edges labeled i correspond to the distinct orbits of i-faces under $G(\mathcal{P})$. Polyhedra with the same symmetry type graph are said to be in the same *class*.

According to [8] and [20] there are three classes of 3-orbit polyhedra, called $3^{0,1}$, 3^1 and $3^{1,2}$ in [8]; they correspond respectively to classes 3^2, 3^{02} and 3^0 in [20]. Their symmetry type graphs are those in Figure 1.

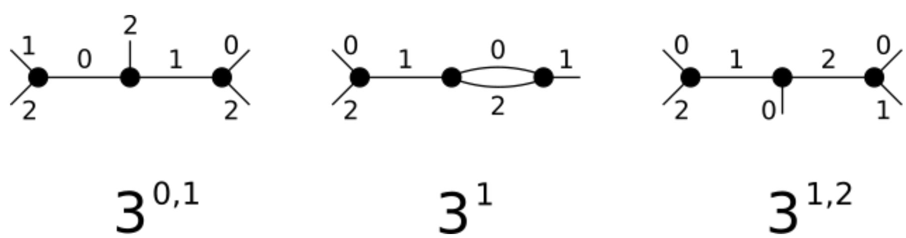


Figure 1: Symmetry type graphs of 3-orbit polyhedra

We will say that an i-face F of a polyhedron $\mathcal{P}$ is 1-*symmetric* if for every $j \in \{0,1,2\}\setminus\{i\}$ there is a symmetry of $\mathcal{P}$ that maps a flag containing F to its j-adjacent flag. In the symmetry type graph of $\mathcal{P}$, 1-symmetric i-faces are those in a flag orbit represented by a vertex with semi-edges with labels in $\{0, 1, 2\} \setminus \{i\}$. In particular, every 3-orbit polyhedron has some 1-symmetric edges; polyhedra in class $3^{0,1}$ have some 1-symmetric vertices; and polyhedra in class $3^{1,2}$ have some 1-symmetric faces.

Under the assumption of trivial flag stabilizers under $G(\mathcal{P})$, the stabilizer of a 1-symmetric i-face F is a dihedral group D_k with $2k$ elements generated by involutions T_j, $j \in \{0,1,2\}\setminus\{i\}$ that fix F as well as the j-face of a given flag containing F. When $i = 1$ then $k = 2$ and hence the edge stabilizer of a 1-symmetric edge is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $i = 0$ then F is a vertex and k is its degree, whereas if $i = 2$ then F is a face and k is its number of edges.

A 1-symmetric face must be a regular polygon; it may be planar or skew. We will write k_p to denote a planar k-gon and k_s to denote a skew k-gon. When k is odd, then a regular

k-gon is necessarily planar, and so we merely denote it by k.

In principle, it is possible to distinguish 1-symmetric faces even more finely by, for example, distinguishing convex polygons from star polygons. However, our notion of *vt-equivalence* that we will develop in Section 4 does not distinguish between the two types of polygons whenever the vertices are 3-valent and the symmetry group contains a plane reflection, which end up accounting for a large proportion of 3-orbit polyhedra in $\mathbb{E}^3$. Thus we will not distinguish between different kinds of planar polygons.

If for every edge e of F there exists $T \in G(\mathcal{P})$ preserving F while interchanging the endpoints of e, but there is no non-trivial symmetry of $\mathcal{P}$ preserving F and fixing one of its vertices, we say that F is 2-*symmetric*. A face of this kind admits a symmetry acting like a 2-step rotation and hence it must have an even number of edges. The symmetries of $\mathcal{P}$ preserving F induce 2 orbits on the flags containing F. Among these symmetries, those that act on F like reflections are illustrated in Figure 2 (a). The 2-symmetric faces of a 3-orbit polyhedron must be vertex-transitive, and they may be planar or skew. As for 1-symmetric faces, we will denote planar and skew faces with a subscript of p or s.

As pointed out in [9], besides the 2-symmetric faces described here, there are other possibilities of polygons and groups acting on them with 2 orbits on the flags, but those scenarios do not appear in the analysis of 3-orbit polyhedra.

We say that a face F is 3-*symmetric* whenever the subgroup of $G(\mathcal{P})$ preserving F induces on it 3 flag orbits. Such faces admit a symmetry of $\mathcal{P}$ acting like a 3-step rotation, but no symmetries acting like a 1- or 2-step rotation. This implies that the number of edges of F is divisible by 3. Furthermore, there are elements of $G(\mathcal{P})$ preserving F that act on it like reflections; some such reflections fix a vertex and some fix midpoints of edges. The symmetries acting like reflections on a 3-symmetric hexagon are illustrated in Figure 2 (b).

The symmetry of a 3-symmetric face that fixes a vertex must either be a plane reflection or a half-turn, while the symmetry that fixes the midpoint of an edge could be a plane reflection, half-turn, or central inversion. Of the six possible cases, we encounter three of them: where both symmetries are plane reflections, where both are half-turns, and where the symmetry that fixes a vertex is a reflection and the other symmetry is a half-turn. We will denote a 3-symmetric k-gon by k_r, k_h, or k_{rh} according to which of these possibilities it realizes. Note that if k is odd, then k_{rh} is not possible. Also, if $k = 3$ then the geometry of the face does not depend on whether the generating symmetry is a reflection or half-turn, and so we merely denote the face by 3.

We extend the definitions of 2-symmetric and 3-symmetric from faces to vertices by duality. In this way, a 2-symmetric vertex v has even degree and there are symmetries of $\mathcal{P}$ that fix v and any of the edges that contain v. On the other hand, 3-symmetric vertices are such that a symmetry of $\mathcal{P}$ acts like a 3-step rotation around them, but no element in $G(\mathcal{P})$ acts like a 1-step or 2-step rotation around them. There are symmetries of $\mathcal{P}$ that fix a 3-symmetric vertex while reversing the cyclic order of the edges around it. The degree of a 3-symmetric vertex is divisible by 3.

Every 3-orbit polyhedron has two orbits on edges. One orbit consists of 1-symmetric

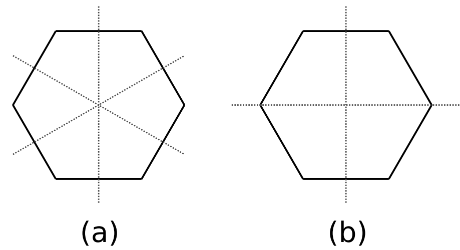


Figure 2: Reflections of 2-symmetric and 3-symmetric hexagons

edges, and the other of edges containing flags in two distinct orbits. If $\mathcal{P}$ is a 3-orbit polyhedron in class $3^{0,1}$ then there is no symmetry swapping the endpoints of a 2-orbit edge, but there is a non-trivial simmetry fixing it pointwise. If $\mathcal{P}$ belongs to either of the other two classes then there are no non-trivial symmetries fixing 2-orbit edges pointwise, but for each edge there is a symmetry that swaps its endpoints. For convenience, we shall refer to 2-orbit edges as 2-symmetric edges regardless of the class of 3-orbit polyhedra in question.

2.1 Class $3^{0,1}$

Polyhedra $\mathcal{P}$ in class $3^{0,1}$ have two orbits of vertices. One orbit contains 1-symmetric vertices, and the other contains 2-symmetric vertices.

These polyhedra also have two orbits of edges. One of these orbits contains 2-symmetric edges joining a 1-symmetric vertex with a 2-symmetric vertex. The stabilizer of an edge e in this orbit has only one non-trivial element; this element swaps the two faces that contain e while fixing both endpoints.

The other orbit of edges contains half as many edges as the previous orbit. These edges are 1-symmetric and join two 2-symmetric vertices. The neighbors of a 2-symmetric vertex v_2 alternate between 1-symmetric and 2-symmetric vertices, implying the following results.

Lemma 2.1. *Every* 2*-symmetric vertex of a polyhedron in class* $3^{0,1}$ *is incident to at least two* 1*-symmetric edges and therefore its degree is an even number* $d \geq 4$*.*

Proposition 2.2. *[9, Proposition 3.3] Every skeletal polyhedron in class* $3^{0,1}$ *must have at least two* 1*-symmetric vertices and three* 2*-symmetric vertices.*

Polyhedra in class $3^{0,1}$ are face-transitive. The faces are 3-symmetric and therefore their number of edges is divisible by 3. One third of the edges in each face are 1-symmetric edges, and two consecutive 1-symmetric edges in a 3-symmetric face are separated by two 2-symmetric edges. The next lemmas are consequences of the symmetries of 3-symmetric polygons.

Lemma 2.3. *Let* F *be a* 3*-symmetric face of a* 3*-orbit polyhedron in class* $3^{0,1}$*, and let* e *be an edge of* F *between two* 2*-symmetric vertices. Let* T *be a nontrivial involutory symmetry* *that fixes F and interchanges the two endpoints of e. Then T does not fix any 2-symmetric vertex of F.*

Proof. Suppose that F is a k-gon with vertices labeled $(1, 2, \ldots, k)$ and suppose that T fixes some vertex of F. Without loss of generality we may assume that T interchanges 1 and k. Then it follows that it interchanges every i with $k + 1 - i$. In particular, if T fixes a vertex, then k must be odd. Furthermore, since the number of 2-symmetric vertices in F must be $2k/3$ and the number of 1-symmetric vertices must be $k/3$, there is an even number of 2-symmetric vertices and an odd number of 1-symmetric vertices. It follows that the vertex fixed by T is 1-symmetric. □

Given a 3-orbit polyhedron $\mathcal{P}$ in class $3^{0,1}$ we may preserve one orbit of vertices, while moving each of the vertices in the other orbit a fixed amount along the line that joins it with the center of $\mathcal{P}$. We say that two polyhedra obtained in this way are *vi-equivalent*. We will study 3-orbit polyhedra in class $3^{0,1}$ up to vi-equivalence.

If $\mathcal{P}$ is in class $3^{0,1}$, then its faces are 3-symmetric a-gons, and it has 1-symmetric b-valent vertices and 2-symmetric c-valent vertices, for some a,b, and c. We note that moving one orbit of vertices may disrupt the planarity of a vertex-figure, so since we classify 3-orbit polyhedra in class $3^{0,1}$ up to vi-equivalence, we do not record whether the vertex-figures are planar or not. As described earlier, we will denote the faces as a_r, a_h, or a_{rh}. We will associate to each polyhedron in this class an extended *Schläfli symbol* like $\{a_r, (b, c)\}$.

2.2 Class 3^1

Polyhedra in class 3^1 are vertex- and face-transitive. Their vertices and faces are 3-symmetric. They can be described by a Schläfl symbol like $\{9_r, 6_h\}$, where for both the faces and vertex-figures, we describe the generating involutions of their symmetry group.

They have two orbits on edges. One edge orbit contains 1-symmetric edges. The stabilizer of an edge e in the other edge orbit only contains the identity element and an element that swaps the two endpoints of e while interchanging the faces containing e.

They have two kinds of Petrie paths. There are some that contain only 2-symmetric edges, and some that alternate 1-symmetric and 2-symmetric edges.

2.3 Class $3^{1,2}$

Polyhedra in class $3^{1,2}$ behave in a dual way to polyhedra in class $3^{0,1}$. They are vertex-transitive and all their vertices are 3-symmetric. They have two orbits on edges. One edge orbit contains 1-symmetric edges. The elements of the stabilizer of an edge e in the other orbit swap the endpoints of e and preserve the two faces containing e.

These polyhedra have two orbits of faces; one containing 1-symmetric faces and one containing 2-symmetric faces. The 1-symmetric edges belong to two 2-symmetric faces, whereas the 2-symmetric edges belong to a 1-symmetric face and to a 2-symmetric face.

The Petrie paths of polyhedra in class $3^{1,2}$ are all 3-symmetric. In each of them, every third edge is 1-symmetric, and two consecutive 1-symmetric edges are separated by two 2-symmetric edges.

If $\mathcal{P}$ is in class $3^{1,2}$, then its faces consist of 1-symmetric a-gons, 2-symmetric b-gons, and c-valent vertices. The vertices are described as in class 3^1. The 1-symmetric faces and 2-symmetric faces are either planar or skew, and thus are denoted by a_p or a_s for example. If a is odd, then the face is necessarily planar and we do not include a subscript. (On the other hand, b must be even, and so we always use its subscript.) We describe such a polyhedron with a Schläfli symbol $\{(a, b), c\}$ with the subscripts mentioned; e.g., $\{(3, 8_s), 6_{rh}\}$.

2.4 General results

Now let us examine in more detail the geometric structure of the symmetry group of a 3-orbit polyhedron. We start with three useful results on the edges of 3-orbit polyhedra.

Lemma 2.4. *Let $\mathcal{P}$ be a 3-orbit polyhedron with trivial flag stabilizers under $G(\mathcal{P})$. Then there exists an edge orbit $\mathcal{O}$ under $G(\mathcal{P})$ such that for each $e \in \mathcal{O}$, the stabilizer in $G(\mathcal{P})$ of e is generated by:*

- *an involution fixing e pointwise that interchanges the two faces containing e,*
- *an involution that interchanges the endpoints of e and preserves the two faces containing e.*

Furthermore, these involutions commute.

Proof. As mentioned above, the three symmetry type graphs of 3-orbit polyhedra indicate the presence of 1-symmetric edges. The symmetry group of such an edge e acts transitively on the four flags containing e, and in particular there are symmetries that a given one of these flags to its 2-adjacent flag and to its 0-adjacent flag. These symmetries act on the vertices and faces incident to e precisely as described in the items of the statement.

The commutativity follows from the facts that for every flag Φ the flags $(\Phi^0)^2$ and $(\Phi^2)^0$ are equal. Indeed, trivial flag stabilizers imply that there is a unique symmetry mapping Φ to $(\Phi^0)^2 = (\Phi^2)^0$. □

Lemma 2.5. *Let F be a face of a 3-orbit polyhedron $\mathcal{P}$ with irreducible symmetry group. Consider a vertex v of F and its neighbors u and w, and suppose that the edge connecting u and v is 1-symmetric. Let T be the nontrivial symmetry of $\mathcal{P}$ that fixes u and v. Then T does not fix w.*

Proof. First, recall that every 1-symmetric edge does have a nontrivial symmetry that fixes u and v. This symmetry cannot fix F; otherwise it would fix a flag, implying that it is the identity. If T fixed w, then F and $T(F)$ would both contain the vertices u, v, and w in the same order. But then the vertex-figure at v would contain a 2-cycle and be disconnected. □

The Orbit-Stabilizer Theorem immediately implies the following:

Proposition 2.6. *Let $\mathcal{P}$ be a 3-orbit polyhedron such that $|G(\mathcal{P})| = N$. Then $\mathcal{P}$ has precisely $N/4$ 1-symmetric edges and $N/2$ 2-symmetric edges.*

The symmetry groups of 3-orbit polyhedra must contain many involutions, as illustrated by the following two results. The first one follows directly from the description of 1-, 2- and 3-symmetric vertices. The second one is a direct consequence of [8, Corollary 2].

Lemma 2.7. *Let $\mathcal{P}$ be a 3-orbit polyhedron and v one of its vertices. Then the stabilizer in $G(\mathcal{P})$ of v contains a non-trivial involution. Furthermore, if $\mathcal{P}$ is not vertex-transitive, then the stabilizer of v is generated by two involutions.*

Proposition 2.8. *If $\mathcal{P}$ is a 3-orbit polyhedron then $G(\mathcal{P})$ is generated by involutions.*

Involutions shall play an important role in our analysis of 3-orbit polyhedra. In that analysis we shall make use of the following obvious remark.

Remark 2.9. *Let F be a 3-symmetric face, and let T be a nontrivial involutory symmetry of F. If T does not fix any vertices of F, then F has an even number of vertices.*

Now we provide a condition on the number of vertices of a 3-orbit polyhedron in terms of the size of its symmetry group.

Proposition 2.10. *If $\mathcal{P}$ is a 3-orbit polyhedron with symmetry group of order N, then the number V of vertices satisfies $V > \sqrt{3N/2}$.*

Proof. A 3-orbit polyhedron with a group of order N has $3N$ flags and $3N/4$ edges. So there must be at least $3N/4$ pairs of vertices, which means $V(V-1)/2 \geq 3N/4$. Then $V(V-1) \geq 3N/2$; in particular $V^2 > 3N/2$ and the result follows. □

In upcoming sections we shall encounter graphs where all the vertices lie on some sphere $\mathcal{S}$. If there are no edges joining antipodal vertices then we may project the edges to $\mathcal{S}$ and obtain an embedding of the graph on the sphere (possibly with edge crossings). In order to describe the faces of a polyhedron with such a 1-skeleton we shall say that the face F *skips m edges* at some vertex v with degree d if in some neighborhood of v on $\mathcal{S}$ the edges of F leave m edges on one side and $d-m-2$ on the other. When doing so, we assume a fixed global orientation of $\mathcal{S}$, so that when tracing the face we always skip edges on the same side.

For example, when describing the great dodecahedron (see [5]) we may say that we consider the 1-skeleton of the icosahedron and build the faces so that they skip an edge at every vertex. We could think of the Petrial of a Platonic solid $\mathcal{P}$ of degree d as the polyhedron with the 1-skeleton of $\mathcal{P}$ where the faces skip alternatingly 0 and $d-2$ edges.

When determining the candidate faces of a polyhedron with a given 1-skeleton we must take into account that the resulting set of faces may not yield a polyhedron because the vertex-figure is not connected (failing the third condition to be a polyhedron). For example, if the 1-skeleton is that of the octahedron then we cannot skip one edge at every vertex, since we would be left with the three equatorial squares; in this situation each vertex-figure is a pair of line segments intersecting only in their midpoints.

2.5 Operations on polyhedra

If $\mathcal{P}$ has a dual, in some cases it is possible to construct a *canonical dual* of $\mathcal{P}$ by taking as vertices of $\mathcal{P}^\delta$ the barycenters of the faces of $\mathcal{P}$. Then the edges and faces are determined by the duality condition. With $\mathcal{P}^\delta$ arising from this construction, $G(\mathcal{P}) = G(\mathcal{P}^\delta)$, and we will simply call $\mathcal{P}^\delta$ 'the dual of $\mathcal{P}$'. However, we may encounter polyhedra where two or more faces have the same barycenter, in which case the resulting structure would not satisfy our definition of polyhedron.

For most of the polyhedra $\mathcal{P}$ we will describe, there are pairs of faces that share more than one edge. Such a polyhedron cannot have a (geometric) dual, since such a dual would have a pair of vertices connected by multiple edges. In fact, if the faces of $\mathcal{P}$ are too big, then no dual can exist:

Proposition 2.11. *Suppose $\mathcal{P}$ is a 3-orbit polyhedron with symmetry group of size N and with a dual. If the faces of $\mathcal{P}$ are all $3k$-gons, then $k < \sqrt{N/6}$.*

Proof. By Proposition 2.10, $\mathcal{P}$ must have more than $\sqrt{3N/2}$ faces. Each face is part of $6k$ flags and there are $3N$ flags total, and the result is simple from there. □

Proposition 2.11 is useful in filling out the tables in Section 14, but we will not explicitly comment on when we are using it.

Note that if $\mathcal{P}$ has a dual $\mathcal{P}^\delta$, then $G(\mathcal{P}) = G(\mathcal{P}^\delta)$. Similarly, if the Petrial of $\mathcal{P}$ ($\mathcal{P}^\pi$) is a polyhedron, then $G(\mathcal{P}) = G(\mathcal{P}^\pi)$.

The three classes of 3-orbit polyhedra are related by duality and Petriality as follows.

Lemma 2.12. *[9, Lemma 3.4] Let $\mathcal{P}$ be a 3-orbit polyhedron having a geometric dual $\mathcal{P}^\delta$.*

- *If $\mathcal{P}$ is in class $3^{0,1}$ then $\mathcal{P}^\delta$ is in class $3^{1,2}$.*
- *If $\mathcal{P}$ is in class 3^1 then $\mathcal{P}^\delta$ is also in class 3^1.*
- *If $\mathcal{P}$ is in class $3^{1,2}$ then $\mathcal{P}^\delta$ is in class $3^{0,1}$.*

Lemma 2.13. *[9, Lemma 3.5] Let $\mathcal{P}$ be a 3-orbit polyhedron whose Petrial $\mathcal{P}^\pi$ is a polyhedron.*

- *If $\mathcal{P}$ is in class $3^{0,1}$ then $\mathcal{P}^\pi$ is also in class $3^{0,1}$.*
- *If $\mathcal{P}$ is in class 3^1 then $\mathcal{P}^\pi$ is in class $3^{1,2}$.*
- *If $\mathcal{P}$ is in class $3^{1,2}$ then $\mathcal{P}^\pi$ is in class 3^1.*

In addition to duality and Petriality, there are a few more operations on polyhedra that play an important role in our classification. The first is truncation. In the convex setting, we may think of 'cutting off' each vertex, obtaining a new polyhedron with two vertices on the relative interior of each original edge, and with two kinds of faces: those that correspond to the original vertex-figures and those that are truncations of the original faces, now with twice the number of edges as the original faces. More generally, truncation (and related operations) can be applied to skeletal polyhedra and indeed to abstract polyhedra; see [26] for a detailed description. The truncation of a regular polyhedron is in most cases a 3-orbit polyhedron, accounting for their prominence in our analysis. If $\mathcal{P}$ is an abstract polyhedron, we will use $\mathrm{Tr}(\mathcal{P})$ to denote the truncation of $\mathcal{P}$. We will also use $T_{[p,q]}$ to denote the 1-skeleton of the (geometric) truncation of $\{p, q\}$.

Dual to truncation is an operation known as the *Kleetope* operation. Applied to convex polyhedra, it can be thought of as attaching a pyramid to each face. For example, the Kleetope of the cube is the *tetrakis hexahedron*, a 3-orbit polytope. For our purposes, we will define the Kleetope operation on abstract polyhedra only, defined by $\mathrm{Kl}(\mathcal{P}) = \mathrm{Tr}(\mathcal{P}^{\delta})^{\delta}$. We will also use $K_{[p,q]}$ to denote the 1-skeleton of the Kleetope of $\{p, q\}$.

The last operation is actually a family of three related operations, and we first describe the effect on a 1-skeleton S. The first operation is called ζ (see [17]), which replaces every edge $\{u, v\}$ with the edges $\{u, -v\}$ and $\{-u, v\}$. Since a 3-orbit polyhedron has two orbits of edges, we may also choose to perform this operation only to the 1-symmetric edges or only to the 2-symmetric edges. We will denote the operation that replaces all i-symmetric edges $\{u, v\}$ with $\{u, -v\}$ and $\{-u, v\}$ by ζ_i. (A word of caution: ζ_i has a different definition in [17] but is used sparingly.) Note that $\zeta = \zeta_1\zeta_2$.

In most cases we encounter, S will be centrally symmetric. However, we will also find it convenient to apply ζ_i to 1-skeleta (and polyhedra) that are not centrally symmetric, such as the truncated tetrahedron.

Definition 2.14. *Let S be the 1-skeleton of a 3-orbit polyhedron with vertex set V and edge set $E_1 \cup E_2$ where E_1 contains the 1-symmetric edges and E_2 contains the 2-symmetric edges. Let $\overline{V} = V \cup -V$, and for $i = 1, 2$ let*

$$\overline{E_i} = \{\{u, -v\} : \{u, v\} \in E_i\}.$$

Then:

(a) *S^{ζ_1} is one connected component of the 1-skeleton $(\overline{V}, \overline{E_1} \cup E_2)$.*

(b) *S^{ζ_2} is one connected component of the 1-skeleton $(\overline{V}, E_1 \cup \overline{E_2})$.*

(c) *S^{ζ} is one connected component of the 1-skeleton $(\overline{V}, \overline{E_1} \cup \overline{E_2})$.*

We note that in the cases where we get a disconnected graph and have to take one connected component, these two components are congruent; in fact, they are the image of one another under a central inversion.

To extend these operations to polyhedra, we need to explain how the faces of $\mathcal{P}^{\zeta_i}$ should be obtained from the faces of $\mathcal{P}$. Informally, the idea is this: start with one typical face from each orbit of faces. We may write the face as a cyclic sequence of vertices $(v_1, v_2, \ldots, v_k)$. Then, depending on which ζ_i we are using and which kinds of edges the face uses, some vertices v_i are replaced with their antipode $-v_i$. In some cases (that will become clear), we need to apply this operation to the walk that goes twice around the face: $(v_1, \dots, v_k, v_1, \dots, v_k)$. In any event, the effect of applying ζ_i now turns our original face into a new cyclic walk $(v'_1, \dots, v'_m)$, with $m = k$ or $m = 2k$. In some cases, we may have $m = k$ but the new cyclic walk actually has period $m/2$, in which case we just consider the new face to be $(v'_1, \dots, v'_{m/2})$. In any case, we define the faces of $\mathcal{P}^{\zeta_i}$ to be the image of such cyclic walks under the symmetry group. We note that it does sometimes happen that the new faces are not polygons, in which case $\mathcal{P}^{\zeta_i}$ is not a polyhedron.

Now let us describe the new faces in more detail.

Definition 2.15. *Let $F = (v_1, \dots, v_k)$ be a face of a 3-orbit polyhedron.*

(a) *If F is 1-symmetric, then all of the edges are 2-symmetric. So ζ_1 fixes F.*

  (a) *If k is even, then F^{ζ_2} is $(v_1, -v_2, v_3, \dots, v_{k-1}, -v_k)$.*

  (b) *If k is odd, then F^{ζ_2} is $(v_1, -v_2, v_3, \dots, -v_{k-1}, v_k, -v_1, v_2, \dots, -v_k)$.*

(b) *If F is 2-symmetric, then the edges alternate between 1-symmetric and 2-symmetric. Without loss of generality, we may assume that the edge from v_i to v_{i+1} is 1-symmetric when i is odd, and 2-symmetric when i is even.*

  (a) *If k is divisible by 4, then F^{ζ_1} is $(v_1, -v_2, -v_3, v_4 \dots, -v_{k-1}, v_k)$ and F^{ζ_2} is $(v_1, v_2, -v_3, -v_4, v_5, \dots, -v_{k-1}, -v_k)$.*

  (b) *If k is not divisible by 4, then F^{ζ_1} is $(v_1, -v_2, -v_3, v_4, \dots, v_{k-1}, -v_k, -v_1, v_2, \dots, v_k)$ and F^{ζ_2} is $(v_1, v_2, -v_3, -v_4, v_5, \dots, v_{k-1}, v_k, -v_1, -v_2, \dots, -v_k)$.*

(c) *If F is 3-symmetric, then without loss of generality, the 1-symmetric edges are $\{v_i, v_{i+1}\}$ with i divisible by 3, and the remaining edges are 2-symmetric.*

  (a) *If k is even (and thus, divisible by 6) then F^{ζ_1} is $(v_1, v_2, v_3, -v_4, -v_5, -v_6, v_7, \dots, -v_k)$.*

  (b) *If k is odd, then F^{ζ_1} is $(v_1, v_2, v_3, -v_4, -v_5, -v_6, v_7, \dots, v_k, -v_1, -v_2, -v_3, \dots, -v_k)$.*

  (c) *For all k, F^{ζ_2} is $(v_1, -v_2, v_3, v_4, -v_5, v_6, \dots, -v_{k-1}, v_k)$.*

We note that, strictly speaking, the description of F^{ζ_i} in Definition 2.15 depends on a particular ordering of the vertices of a face, but since we are really only interested in how ζ_i acts on the whole orbit of faces, it does no harm to make an arbitrary choice of how we describe an individual face.

The action of ζ_2 on 1-symmetric faces has the same effect on them as applying ζ to a regular polygon, which is described in [17, Thm. 5.3]. In our analysis, the 1-symmetric faces are typically either planar faces whose affine hull does not contain the center of the polyhedron, or skew faces where the center of the face coincides with the center of the polyhedron, and ζ_2 interchanges these two possibilities.

In a similar way, both operations ζ_i will transform a planar 2-symmetric face whose affine hull does not contain the center of the polyhedron into a skew face whose center coincides with the center of the polyhedron.

Now suppose F is a 3-symmetric face, and fix a 1-symmetric edge e_1. Consider the symmetry T_1 of F that interchanges the endpoints of e_1. Then if $\mathcal{P}^{\zeta_2}$ is a polyhedron, the same symmetry T_1 fixes the induced face F^{ζ_2} and interchanges the endpoints of e_1. Similarly, if v_1 is a 1-symmetric vertex of F, there is a symmetry T_2 of F that fixes v_1 while interchanging its neighbors, and the same symmetry will act in the same way on F^{ζ_2}. Thus we see that the symmetry group of a 3-symmetric face remains unchanged under ζ_2, and so the size and the type of the face (n_r, n_h, or n_{rh}) remain unchanged as well.

We can now extend these operations to polyhedra.

Definition 2.16. *Let P be a 3-orbit polyhedron. For $i = 1, 2$ we define a new structure P^{ζ_i} (which may or may not be a polyhedron) as follows:*

(a) *If S is the 1-skeleton of P, then the 1-skeleton of P^{ζ_i} will be S^{ζ_i}.*

(b) *The faces of P^{ζ_i} are obtained from the faces of P in the manner described in Definition 2.15. We keep only those faces that lie in the connected component of S^{ζ_i} that we chose.*

Let us note that if $\mathcal{P}$ is vertex-intransitive, then $\mathcal{P}^{\zeta_2}$ is vi-equivalent to $\mathcal{P}$, since up to vi-equivalence we may scale one orbit of vertices independently of the other and so we can send, say, every 1-symmetric vertex v to $-v$. Thus, we will only use ζ_2 on vertex-transitive polyhedra.

Example 2.17. *Let P be the truncated cube (Figure 3a). Then the 1-skeleton of P^{ζ_1} has two connected components, each containing 12 vertices (Figure 3b). The 1-skeleton of P^{ζ_2} is connected, and the faces are skew hexagons and skew octagons (Figure 3c where only one 2-symmetric face is shown).*

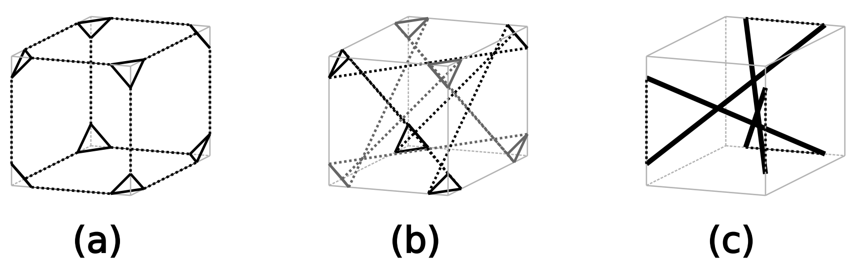


Figure 3: The truncated cube $\mathrm{Tr}(\{4,3\})$ together with $\mathrm{Tr}(\{4,3\})^{\zeta_1}$ and $\mathrm{Tr}(\{4,3\})^{\zeta_2}$; 1-symmetric edges are dotted and 2-symmetric edges are solid

3 Finite irreducible groups of isometries of $\mathbb{E}^3$

In this section we recall the finite irreducible groups of isometries of $\mathbb{E}^3$. Throughout we follow [12], although with a different notation.

All finite groups of isometries have a fixed point. A finite group G of isometries of $\mathbb{E}^3$ is said to be *affinely reducible* whenever there is a setwise fixed line, and hence also a setwise fixed plane (the orthogonal complement of the fixed line through the fixed point). Finite groups of isometries that are not affinely reducible are called *affinely irreducible*.

There are seven finite affinely irreducible groups of isometries of $\mathbb{E}^3$. They are tightly linked with the symmetry groups of the Platonic solids, and are described next.

The *rotational tetrahedral group* $[3,3]^+$ is the group of orientation preserving symmetries of the tetrahedron. It contains 12 elements and can be understood as the alternating group on the four vertices of the tetrahedron.

The *full tetrahedral group* $[3,3]$ is the group of all symmetries of the tetrahedron and contains 24 elements. It can be identified with the group of permutations of the vertices of the tetrahedron, and hence it is isomorphic to the symmetric group S_4.

The *rotational octahedral group* $[3,4]^+$ is the group of orientation preserving symmetries of the cube (and of the octahedron). It contains 24 elements and can be understood as the symmetric group on the four main diagonals of the cube.

The *full octahedral group* $[3,4]$ is the group of all symmetries of the cube and contains 48 elements. It is isomorphic to the direct product $S_4 \times C_2$ of the symmetric group on 4 elements with a cyclic group of order 2. The central element of $[3,4]$ is the central inversion with respect to the center of the cube.

The full octahedral group contains two index-two subgroups besides the rotational octahedral group. They can be understood through the two vertex-disjoint tetrahedra inscribed in the cube. In particular, the full tetrahedral group is the index-two subgroup of the full octahedral group consisting of all isometries preserving the two inscribed tetrahedra of the cube.

The group denoted by $[3,3]^*$ in [24] is the group of all orientation preserving isometries that fix the two tetrahedra of the cube together with all orientation reversing isometries that interchange these two tetrahedra. Since it includes the central inversion with respect to the center of the cube, it is isomorphic to $A_4 \times C_2$.

The *rotational icosahedral group* $[3,5]^+$ is the group of orientation preserving symmetries of the dodecahedron (and of the icosahedron). It contains 60 elements and can be understood as the alternating group on the five cubes inscribed in the dodecahedron.

Finally, the *full icosahedral group* $[3,5]$ is the group of all symmetries of the dodecahedron and contains 120 elements. It is isomorphic to the direct product $A_5 \times C_2$ of the alternating group on 5 elements with a cyclic group of order 2. The central element of $[3,5]$ is the central inversion with respect to the center of the dodecahedron.

No polyhedron with symmetry group $[3,4]^+$ or $[3,5]^+$ is invariant under orientation reversing isometries. It follows that all polyhedra with those symmetry groups are *handed* in the sense that a right polyhedron cannot be overlapped into a left polyhedron through a continuous motion. Nevertheless, right and left handed versions of the same polyhedron are similar to each other, and such a polyhedron appears only once in our enumeration.

The following lemma will be used in subsequent sections. It can be proven by direct inspection.

Lemma 3.1. *Let $G \in \{[3,3],[3,4],[3,5]\}$ and $T \in G$ of order $m > 2$. Then no power of T is a plane reflection.*

4 Affinely irreducible 3-orbit polyhedra

In the following sections we shall enumerate all 3-orbit polyhedra $\mathcal{P}$ in $\mathbb{E}^3$ where $G(\mathcal{P})$ is irreducible. Since the vertex set spans the entire 3-dimensional space, the flag stabilizers in $G(\mathcal{P})$ are trivial.

4.1 General results on symmetry groups of 3-orbit polyhedra

Not every finite irreducible group of isometries of $\mathbb{E}^3$ is the symmetry group of a 3-orbit polyhedron.

Theorem 4.1. *There are no 3-orbit polyhedra $\mathcal{P}$ in $\mathbb{E}^3$ whose symmetry group $G(\mathcal{P})$ is either $[3,3]^+$ or $[3,3]^*$.*

Proof. This follows from Proposition 2.8, since neither $[3,3]^+$ nor $[3,3]^*$ is generated by its set of involutions. To see this, note that all involutions in these two groups fix each coordinate axis setwise. □

Furthermore, not every finite irreducible group that remains is the symmetry group of a vertex-intransitive 3-orbit polyhedron.

Theorem 4.2. *There are no 3-orbit polyhedra in class $3^{0,1}$ whose symmetry group is $[3,4]^+$ or $[3,5]^+$.*

Proof. By Lemma 2.7, each vertex of a polyhedron in class $3^{0,1}$ has a stabilizer that is generated by two involutions. When the symmetry group is $[3,4]^+$ or $[3,5]^+$, the only involutions are half-turns, and the only point that is stabilized by distinct half-turns is the center of the polyhedron, contradicting Proposition 2.2. □

Let us discuss our general approach in the following sections. In each section, we will focus on a particular group G. We start by including a table with information about the group, which looks like the following:

Information about $[3,3]$	
Description	Symmetry group of a tetrahedron $\mathcal{T}$
Order	24
Admissible vertex orbits	4, 6, 12
Involutions	6 plane reflections 3 half-turns whose mirrors join midpoints of edges of $\mathcal{T}$

Since the number of vertices in each orbit must be larger than 1 (by Proposition 2.2) and since each vertex is fixed by some involution (see Lemma 2.7), the row 'admissible vertex orbits' only includes sizes of orbits that meet those restrictions.

For $G \in \{[3,3],[3,4],[3,5]\}$, where there is the possibility of polyhedra in class $3^{0,1}$, we then include a table the summarizes the basic combinatorics that are possible for such polyhedra:

Vertex-intransitive polyhedra with group $[3,3]$	
1-symmetric edges	6
2-symmetric edges	12
1-symmetric vertices	four 3-valent
2-symmetric vertices	four 6-valent or six 4-valent

Similarly, for vertex-transitive polyhedra, we start with the following table, which uses Proposition 2.6, Lemma 2.7, and Proposition 2.10:

Vertex-transitive polyhedra with group $[3,3]$	
1-symmetric edges	6
2-symmetric edges	12
Vertices	12 3-valent

In both cases, we are sometimes able to rule out other sizes of vertex orbit using further geometric arguments. Then we identify the possible 1-symmetric and 2-symmetric edges by considering the sizes of orbits of pairs of points, and considering that we need the 1-skeleton to be connected. Finally, once we have identified a possible 1-skeleton, we determine the possible faces, and then check which choices of faces give us connected vertex-figures.

Some of the 1-skeleta that arise are *cloned* graphs. We recall that the cloned graph of a graph with vertex and edge sets $\{v_1,\dots,v_k\}$ and A, respectively, has as vertex set $\{v_1,\dots,v_k\} \cup \{v'_1,\dots,v'_k\}$ and as edge set $A \cup \{v'w : vw \in A\}$. We can similarly define a cloned 1-skeleton by taking an existing 1-skeleton and then making a copy of the vertices, but dilated so that the new vertices do not overlap the old ones; then we can add the line segments necessary so that the graph of the new 1-skeleton is a cloned graph. We will use $Cl_{[p,q]}$ to denote the cloned 1-skeleton of the polyhedron $\{p,q\}$.

When studying vertex-transitive polyhedra $\mathcal{P}$ with $G(\mathcal{P}) \in \{[3,4],[3,5]\}$ we may prefer to visualize them as bidimensional objects. The vertex-set is contained in a sphere $\mathbb{S}^2$, and the edges may be projected to geodesics of $\mathbb{S}^2$, as long as they do not join antipodal points. The combinatorics of $\mathcal{P}$ can then be completely described in terms of the set of points and projections of edges in $\mathbb{S}^2$, by establishing which cycles correspond to faces.

This visualization is particularly useful when $\mathcal{P}$ is centrally symmetric, since then we can project its immersion to $\mathbb{S}^2$ into the projective plane $\mathbb{P}^2$. There we represent the image of $\mathcal{P}$ under the projection as a set of points with a set of geodesics between them representing edges, and indicating which of those edges are in the same face. As we shall see, the projections of k-gonal faces in $\mathbb{S}^2$ may transform into k-gonal polygons, $(k/2)$-gonal polygons, or degenerate polygons where some vertices (but not all) are visited twice, when projected to $\mathbb{P}^2$.

For our analysis we will search for all i-symmetric polygons ($i \in \{1, 2, 3\}$) with perhaps multiple vertices in $\mathbb{P}^2$ in some given graphs with preestablished 1- and 2-symmetric edges. Each i-symmetric edge has two distinct lifts from $\mathbb{P}^2$ to $\mathbb{S}^2$ and so each 1-symmetric polygon has two distinct lifts to a polygon in $\mathbb{S}^2$, while 2- and 3-symmetric polygons have each 4 lifts to $\mathbb{S}^2$.

4.2 Vertex-transitive polyhedra with non-rigid vertex sets

In the enumeration of 3-orbit polyhedra we shall describe several vertex-transitive polyhedra whose vertices have one degree of freedom (up to similarity). Here we develop the terminology and strategy to follow for such cases. We start by determining when this occurs.

Proposition 4.3. *Let $\mathcal{P}$ be a vertex-transitive 3-orbit polyhedron with 3-valent vertices such that $G(\mathcal{P}) \in \{[3, 3], [3, 4], [3, 5]\}$. Then*

(a) *The stabilizer of every vertex is of order 2 and is generated by a plane reflection.*

(b) *The non-trivial element in the pointwise stabilizer of each 1-symmetric edge is a plane reflection.*

Proof. Let v be a vertex of $\mathcal{P}$. From Lemma 2.7 we know that there is a non-trivial involution T fixing v. Furthermore, since v is contained in 3 edges, it is contained in 6 flags. Vertex-transitivity of $\mathcal{P}$ forces the stabilizer of v in G to consist only of T and the identity element. Besides plane reflections, the involutions in G are half-turns, and the central inversion if $G \neq [3, 3]$. The axis of every half-turn is contained in a plane reflection, implying that T cannot be a half-turn, since otherwise the stabilizer of v would have more than 2 elements. On the other hand, the central inversion only fixes the center of $\mathcal{P}$, and hence it cannot stabilize v. This proves the first item.

The second item follows from the first, since the pointwise stabilizer of an edge must also stabilize both its vertices. □

The plane reflection in Proposition 4.3 (a) can be defined also in the following alternative way.

Lemma 4.4. *Let $\mathcal{P}$ be a vertex-transitive 3-orbit polyhedron such that each vertex stabilizer is generated by a plane reflection. Let u, v, w be vertices of $\mathcal{P}$ such that there are 2-symmetric edges $\{u, v\}$ and $\{v, w\}$. Then the reflection plane of the generator of the stabilizer of v is the bisector of the line segment between the vertices u and w.*

Proof. We know that there is a symmetry of $\mathcal{P}$ fixing v while swapping its two 2-symmetric edges. This must be the only non-trivial element in the stabilizer of v, which by hypothesis is a plane reflection. In order to interchange u and w, the fixed set of the reflection must be the bisector of the line segment between the vertices u, w. □

Proposition 4.3 characterizes the vertex-transitive 3-orbit polyhedra with irreducible symmetry group where each vertex stabilizer is generated by a plane reflection. Indeed, if $\mathcal{P}$ is such a polyhedron then $G(\mathcal{P})$ must contain plane reflections and therefore $G(\mathcal{P}) \in \{[3, 3], [3, 4], [3, 5]\}$. Vertex-transitivity forces the number of vertices to be $|G(\mathcal{P})|/2$, and it follows from Proposition 2.6 that every vertex has degree 3. The vertices of such polyhedra have one degree of freedom in the sense explained below.

We may codify the combinatorics of $\mathcal{P}$ through the action of $G(\mathcal{P})$ on the vertex set as follows. The vertices of $\mathcal{P}$ are $v_0G(\mathcal{P})$, for any given vertex v_0 of $\mathcal{P}$. Each edge may be interpreted as its pair of endpoints, written in the form $\{v_0T_1, v_0T_2\}$ for some $T_1, T_2 \in G(\mathcal{P})$. Finally, each face may be written as its sequence of vertices $(v_0T_1, v_0T_2, \ldots, v_0T_k)$. In this way, the vertex set consists of a list of images of v_0 under elements of $G(\mathcal{P})$ (the orbit of v_0), the edge set consists of pairs of elements of $G(\mathcal{P})$ applied to v_0, and the set of faces consists of vectors of elements of $G(\mathcal{P})$ applied to v_0.

Let R be the generator of the stabilizer of v_0 and let Π be its fixed set. If we replace the base vertex v_0 by some other point w_0 in Π such that its stabilizer under $G(\mathcal{P})$ is also $\langle R \rangle$ then we may use the codification above to recover a polyhedron $\mathcal{P}_w$ combinatorially isomorphic to $\mathcal{P}$. The vertex set of $\mathcal{P}_w$ is $w_0G(\mathcal{P})$. The edge set is now

$$\{\{w_0T_1, w_0T_2\} \,:\, \{v_0T_1, v_0T_2\} \text{ is an edge of } \mathcal{P}\},$$

and the set of faces is

$$\{(w_0T_1, \ldots, w_0T_k) \,:\, (v_0T_1, \ldots, v_0T_k) \text{ is a face of } \mathcal{P}\}.$$

By following the convention that the distance from a vertex to the center o of $\mathcal{P}$ is some constant r, we are in practice admitting vertices in the intersection of Π with the sphere $S(o, r)$ centered at o of radius r. Clearly, a small movement of the vertex is reflected as a small movement of the entire structure, and so this can be understood as a continuous family of realizations of the same combinatorial polyhedron. Two polyhedra obtained in this fashion one from the other will be said to be *vt-equivalent* (the 'vt' standing for 'vertex-transitive').

When choosing the vertex w_0 in Π we required that its stabilizer under $G(\mathcal{P})$ is $\langle R \rangle$. If the stabilizer is larger then the number of vertices will be smaller, implying that the resulting structure is no longer combinatorially isomorphic to $\mathcal{P}$. Therefore, vt-equivalent polyhedra cannot be taken continuously along all of $\Pi \cap S(o,r)$; in general there will be a finite number of points where the resulting structure is not isomorphic to the starting polyhedron.

Figure 4 illustrates the concept of vt-equivalence. We take as example the truncated cube, as shown in (b); the black dotted lines are the 1-symmetric edges; the solid black lines are the edges of the base face, and the dark gray solid lines are the edges of the remaining 7 triangles. The base vertex v_0 is assumed to be in the reflection plane shown in (a), and so it is allowed to belong to either one of two antipodal pair of edges, or to one of two antipodal diagonals of squares. The thin dotted line in (a) indicates the intersection with the upper face of the underlying cube of some reflection plane Δ, defined so that one of the neighbors of v_0 by a 2-symmetric edge is its image under the reflection about Δ. Figure (c) shows the base triangle f_0 together with the three triangles that are joined to f_0 by 1-symmetric edges (again in black dotted lines). This realization can be thought of as flipping the two ends of each 1-symmetric edge from (b), and carrying the triangles along while flipping the vertices. Finally, (d) shows a realization where the base vertex is chosen in a diagonal of square. The black triangle is again the base triangle f_0; for convenience, the diagonals of the squares where the vertices of f_0 lie are also indicated in that diagram. The three triangles joined to f_0 by 1-symmetric edges (thin dotted lines) are shown in dark gray lines. The convex hull of the vertex set of the realizations where v_0 is in a diagonal of a square is a (possibly non-Archimedean) rhombicuboctahedron.

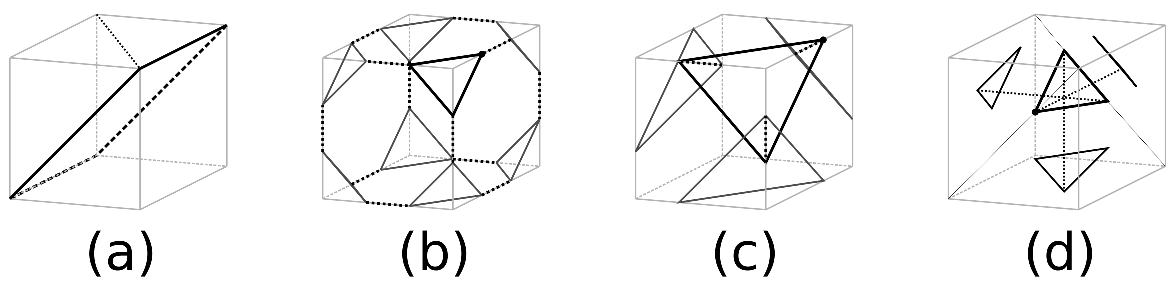


Figure 4: Vt-equivalent realizations of the truncated cube

The previous discussion can be understood as a variation of Wythoff's construction, described in [5, Section 5.7]. In that book it is devoted to polyhedra, and polytopes in general, constructed from groups generated by reflections. Later it was expanded for more general polyhedra (see for example [6], [24] and [25]).

4.3 Vertex-transitive polyhedra with 3-valent vertices

The aim of this subsection is to simplify the enumeration of vertex-transitive 3-valent polyhedra. Our first step is to show that the enumeration of vertex-transitive 3-valent polyhedra in class 3^1 can be obtained from that of polyhedra in class $3^{1,2}$. That result is preceded by the following technical lemma.

Lemma 4.5. *Let $\mathcal{P}$ be a vertex-transitive 3-orbit polyhedron with 3-valent vertices and such that $G(\mathcal{P}) \in \{[3,3],[3,4],[3,5]\}$. Then $\mathcal{P}$ is vt-equivalent to a polyhedron whose 1-symmetric edges do not contain its center.*

Proof. Let e be a 1-symmetric edge and Π the reflection plane of the non-trivial symmetry of $\mathcal{P}$ that stabilizes e pointwise (see Proposition 4.3).

Assume that the 1-symmetric edges of $\mathcal{P}$ contain the center o of $\mathcal{P}$. Then there exists $R \in G(\mathcal{P})$ that swaps the endpoints of e and that it is not the central inversion. In this situation the fixed set of R intersects Π in a line m, and e is contained in the perpendicular m' to m through o in Π. Any valid choice of v_0 in $(S(o,r) \cap \Pi) \setminus m$ (see Subsection 4.2) yields a polyhedron vt-equivalent to $\mathcal{P}$, but whose 1-symmetric edges do not contain o. □

Proposition 4.6. *Let $\mathcal{P}$ be a vertex-transitive finite 3-orbit polyhedron in $\mathbb{E}^3$ with 3-valent vertices and symmetry group $[3,3]$, $[3,4]$ or $[3,5]$. Then $\mathcal{P}^\pi$ is a polyhedron.*

Proof. We only need to show that all Petrie paths are polygons, since the 1-skeleta and vertex-figures of $\mathcal{P}$ and $\mathcal{P}^\pi$ are the same.

If some Petrie path $\pi = (x_1, \dots, x_m, x_1)$ repeats some vertex v then it must repeat at least one of the three edges incident to v. Suppose that $\{v,w\}$ is such an edge. Given that $\mathcal{P}$ is either in class 3^1 or in class $3^{1,2}$, its Petrie paths can be 1-, 2-, or 3-symmetric polygons as defined in Section 2. We shall split the discussion according to the kind of symmetry of π.

Suppose first that π is 1-symmetric and let S be the 1-step rotation along π. Then S has order greater than 2 and therefore it must be either a rotation or a rotatory reflection. In either case, the vertex set of π is $v\langle S\rangle$ and the edge set is $\{v,w\}\langle S\rangle$ implying that π is a polygon.

Now suppose that π is 2-symmetric and let S be the 2-step rotation along π. In this situation the vertex set is $\{v,w\}\langle S\rangle$ and the edge set is $\{\{v,w\},\{w,vS\}\}\langle S\rangle$ (here we are assuming that S rotates v two steps in the direction of its neighbor w). The only way to have a repetition of v in π is if $v = wS^j$ for some $j \in \{1,\dots,m-1\}$, but in that case vertices become 4-valent which contradict our hypothesis. (For example, v would be adjacent to w, to wS^{-1}, to vS^j and to vS^{j+1}.)

Finally, suppose that π is 3-symmetric and let S be the 3-step rotation along π. There are two possible ways on which the edge $\{v,w\}$ repeats depending on the order on which the vertices v,w appear in π:

(A) $(\dots,v,w,\dots,v,w,\dots)$, or

(B) $(\dots,v,w,\dots,w,v,\dots)$.

These possibilities are illustrated in Figure 5, where 1-symmetric edges are indicated in gray.

If the repetition happens as in (A), then $\{v,w\}$ cannot be a 1-symmetric edge (see Figure 5 (b)), since otherwise a power of S would take one appearance of $\{v,w\}$ to the other while

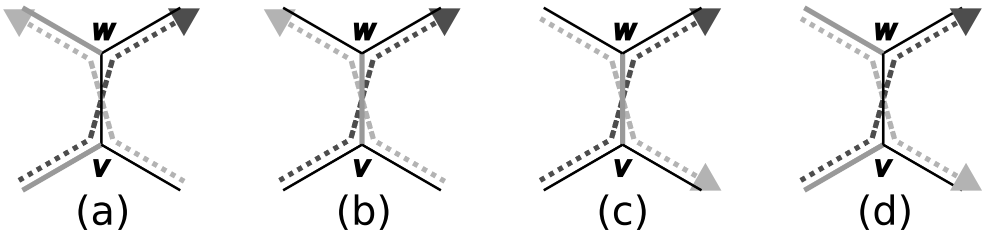


Figure 5: Possibilities of repetition of an edge in π (gray edges are 1-symmetric)

preserving π. This would force the non-trivial pointwise stabilizer of $\{v, w\}$ to be a power of S, contradicting Lemma 3.1.

We next assume still that the repetition happens as in (A), and that $\{v, w\}$ is a 2-symmetric edge (see Figure 5 (a)). In this situation precisely one of the edges at v in π must be 1-symmetric (or a power of S would take one appearance of $\{v, w\}$ to the other while preserving π contradicting that the pointwise stabilizer of a 2-symmetric edge is trivial), implying that at the other spot v appears between two 2-symmetric edges. Without loss of generality, assume that in its first appearance v is incident to a 1-symmetric edge and in its second one it is incident to two 2-symmetric edges. Then for some k the first appearance of w is mapped by S^k to the second appearance of v while preserving π. Furthermore, there exists $T \in \mathcal{P}$ that preserves π and v while interchanging its two neighbors in its second appearance (since it is incident to two 2-symmetric edges). It follows that $S^kT \in \mathcal{P}$ swaps the vertices of the 2-symmetric edge $\{v, w\}$ while fixing π, a contradiction to the fact that the symmetry group of π induces 3 orbits on its flags.

We are left with the case where π is 3-symmetric and the repetition occurs as in (B). We can discard the possibility of $\{v, w\}$ being 2-symmetric (see Figure 5 (d)) as in the case where the repetition is as in (A). The only differences is that here it suffices a power of S (the symmetry T in the previous case plays no role). Therefore we may assume that $\{v, w\}$ is 1-symmetric (see Figure 5 (c)).

If $\{v, w\}$ is 1-symmetric and π does not repeat any 2-symmetric edge then $\{v, w\}S^{m/6} = \{v, w\}$ (the order of S is $m/3$ and $\{v, w\}$ repeats precisely twice). It follows that the order of S is even. The order of S cannot be 2, since otherwise π would be a hexagon of the form (v, w, y, w, v, z), and consecutive edges $\{w, y\}$ and $\{y, w\}$ are not possible in Petrie paths. The symmetry S cannot be a rotatory reflection whose order is twice an odd number, since otherwise $S^{m/6}$ is the central inversion with respect to the center o of $\mathcal{P}$ (this is the case for all such rotatory reflections in G), implying that the midpoint of the edge $\{v, w\}$ contains o, which by Lemma 4.5 we may assume not to be true.

The only remaining cases are when G is either $[3, 3]$ or $[3, 4]$, and S is a rotation or rotatory reflection of order 4. In that situation S^2 is a half-turn and its axis must contain the midpoints of the two 1-symmetric edges of π. Since $[3, 3]$ is a subgroup of $[3, 4]$, we may assume that the two 1-symmetric edges of π lie on the two lines shown in either of the cubes in Figure 6. Furthermore, if w is a vertex of π that is not incident to a 1-symmetric

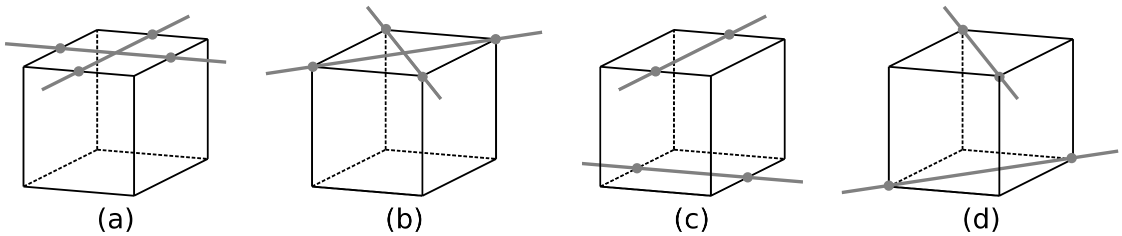


Figure 6: Possibilities for the lines containing 1-symmetric edges of π

edge of π then by Lemma 4.4 the stabilizer of w must be the bisector of the line segment between its two neighbors u and v. Such a bisector is not a plane reflection of $[3,4]$ if the 1-symmetric edges are in the lines in Figure 6 (c) or (d), since neither u nor v are in the intersection of two reflection planes in $[3,4]$. On the other hand, if the 1-symmetric edges are in the lines in Figure 6 (a) or (b) then w lies on the plane containing two opposite vertical edges of the cube, or two opposite vertical altitudes of squares of the cube. In either situation, vertex-transitivity of $\mathcal{P}$ forces the existence of an element in $[3,4]$ mapping the reflection plane containing w to that containing u, but these two planes make an angle of $\pi/4$, a contradiction. □

Corollary 4.7. *The enumerations of all finite polyhedra with* 3*-valent vertices in class* 3^1 *with symmetry groups* $[3,3]$, $[3,4]$ *and* $[3,5]$ *consist of the Petrials of all finite polyhedra with* 3*-valent vertices and with those symmetry groups in class* $3^{1,2}$*.*

In view of Corollary 4.7 we may restrict our attention to polyhedra in class $3^{1,2}$. One kind of 3-orbit polyhedron in that class that frequently occurs is truncations of regular polyhedra. In what follows we shall show that every 3-orbit polyhedra in class $3^{1,2}$ is vt-equivalent to either the truncation of a regular polyhedron, or to its image under the operation ζ_2. Let $\mathcal{P}$ be a polyhedron in class $3^{1,2}$ with 3-valent vertices and symmetry group in $\{[3,3],[3,4],[3,5]\}$.

We continue the analysis of the position of the edges of $\mathcal{P}$ relative to the center of $\mathcal{P}$.

Lemma 4.8. *Let* $\mathcal{P}$ *be a vertex-transitive* 3*-orbit polyhedron in class* $3^{1,2}$ *with* 3*-valent vertices and such that* $G(\mathcal{P}) \in \{[3,3],[3,4],[3,5]\}$*. Then the* 2*-symmetric edges do not contain the center of* $\mathcal{P}$*.*

Proof. Let f_1 be a 1-symmetric face containing the base vertex v_0. Then there exists a rotation or rotatory reflection $R \in G(\mathcal{P})$ whose order is at least 3, and acts like a 1-step rotation on the (2-symmetric) edges of f_1. We can conclude that the 2-symmetric edges do not intersect the center of $\mathcal{P}$, since otherwise it would not be possible to form f_1 from the orbit of such edges under $\langle R\rangle$. □

It can be shown that the conclusion of Lemma 4.8 is also true for 1-symmetric edges when $G(\mathcal{P}) \in \{[3,4],[3,5]\}$. However, if $G(\mathcal{P}) = [3,3]$ there is a particular choice of v_0 where the 1-symmetric edges do contain the center of $\mathcal{P}$. For our purposes we only need Lemma 4.5.

The next proposition is a direct consequence of Lemmas 4.8 and 4.5.

Proposition 4.9. *Let $\mathcal{P}$ be a vertex-transitive 3-orbit polyhedron in class $3^{1,2}$ with 3-valent vertices and such that $G(\mathcal{P}) \in \{[3,3],[3,4],[3,5]\}$. Then $\mathcal{P}$ is vt-equivalent to a polyhedron, none of whose edges contains the center of $\mathcal{P}$.*

We next analyze the 1-symmetric faces of $\mathcal{P}$. Each of them must be invariant under a rotation or a rotatory reflection of order at least 3, that permutes its edges cyclically.

Lemma 4.10. *Let $\mathcal{P}$ be a polyhedron in class $3^{1,2}$ with 3-valent vertices and symmetry group in $\{[3,3],[3,4],[3,5]\}$. Let f be a 1-symmetric face of $\mathcal{P}$, and let $R \in G(\mathcal{P})$ of order at least 3 be such that it cyclically permutes the edges of f.*

(a) *If R is a rotatory reflection then $\mathcal{P}$ is vt-equivalent to a polyhedron with skew 1-symmetric faces.*

(b) *If R is a rotation then $\mathcal{P}$ is vt-equivalent to a polyhedron whose center is not in the affine span of any of its 1-symmetric faces.*

Proof. Let v be a vertex of f and let Π the reflection plane of $G(\mathcal{P})$ containing v. Then the reflection about Π preserves f, since it is the only non-trivial symmetry of $\mathcal{P}$ that fixes v and f is 1-symmetric. It follows that the axis ℓ of R is contained in Π.

If R is a rotatory reflection and f is planar or R is a rotation and the affine span of f contains the center of $\mathcal{P}$ then v is in the perpendicular of ℓ at the center of $\mathcal{P}$ in Π, and we may choose v outside that perpendicular to obtain another polyhedron $\mathcal{Q}$ vt-equivalent to $\mathcal{P}$. If R is a rotatory reflection, $\mathcal{Q}$ has skew 1-symmetric faces; if R is a rotation then the affine span of the 1-symmetric faces of $\mathcal{Q}$ does not contain the center of $\mathcal{Q}$. □

In view of Proposition 4.9 and Lemma 4.10 we say that a vertex-transitive 3-orbit polyhedron $\mathcal{P}$ in class $3^{1,2}$ with 3-valent vertices and symmetry group in $\{[3,3],[3,4],[3,5]\}$ is in *general position* whenever

- none of its edges contains the center of $\mathcal{P}$,
- either $\mathcal{P}$ has skew 1-symmetric faces, or it has planar 1-symmetric faces whose affine span do not contain the center of $\mathcal{P}$.

In order to relate polyhedra with planar 1-symmetric faces with polyhedra where they are non-planar we need the following result.

Proposition 4.11. *Let $\mathcal{P}$ be a 3-orbit polyhedron in class $3^{1,2}$ in general position whose vertices are 3-valent, whose 1-symmetric faces are non-planar (resp. planar), and whose symmetry group is in $\{[3,3],[3,4],[3,5]\}$. Then $\mathcal{P}^{\zeta_2}$ is a 3-orbit polyhedron in class $3^{1,2}$, and its 1-symmetric faces are planar (resp. non-planar).*

Proof. By construction, $\mathcal{P}^{\zeta_2}$ has 1-symmetric and 2-symmetric faces; hence it is in class $3^{1,2}$.

Now, if the 1-symmetric faces of $\mathcal{P}$ lie on a plane Λ then each 2-symmetric edge of $\mathcal{P}^{\zeta_2}$ has one endpoint in Λ and the other in the image of Λ under the central inversion at the center of $\mathcal{P}$ (which is not equal to Λ, by general position). Hence, the 1-symmetric faces of $\mathcal{P}^{\zeta_2}$ are skew. Similarly, if the 1-symmetric faces of $\mathcal{P}$ lie on two parallel planes (image one of the other under the central inversion) then the 1-symmetric faces of $\mathcal{P}^{\zeta_2}$ are planar.

Suppose that we can show that the 2-symmetric faces are polygons. Since the 1-symmetric faces of $\mathcal{P}^{\zeta_2}$ are always polygons, we have that all its faces are indeed polygons. By definition, $\mathcal{P}^{\zeta_2}$ satisfies the first two items of our definition of polyhedron. Furthermore, each vertex v lies on the plane Π fixing the 1-symmetric edge at v, and the reflection about that plane swaps the two 2-symmetric edges at v. It follows that in $\mathcal{P}^{\zeta_2}$ the 1-symmetric edge at v stays in Π while the two other neighbors of w do not belong to that plane. From this we conclude that the vertex-figures are triangles, and hence $\mathcal{P}^{\zeta_2}$ is a polyhedron.

It remains to prove that the 2-symmetric faces are polygons. Let F be a 2-symmetric face of $\mathcal{P}$. If $F = (v_1, \dots, v_k)$ then $F^{\zeta_2} = (v_1, v_2, -v_3, -v_4, v_5, ...)$, assuming that the 2-symmetric edges are the edges from v_i to v_{i+1} when i is even. The only way that F^{ζ_2} can fail to be a polygon is if some vertex is repeated and incident to two distinct 2-symmetric edges.

So, suppose that F^{ζ_2} has 2-symmetric edges $\{u, v\}$ and $\{u, w\}$ with $v \neq w$. Then there must have been 2-symmetric edges $\{u, -v\}$ and $\{u, -w\}$ among the edges of F and $-F$, and since these were both polygons, it follows that F contains just one of those edges; say $\{u, -v\}$. (In particular, this implies that $F \neq -F$.) Then F and $-F$ have the vertex u in common. Since F is vertex-transitive and the central involution commutes with all symmetries of F, it follows that F and $-F$ have all of their vertices in common. Now, since $\mathcal{P}$ is 3-valent, every vertex is incident to a unique 1-symmetric edge. Thus, since F and $-F$ share all of their vertices, they also share all of their 1-symmetric edges.

Consider a vertex v of F and let $\{u, v\}$ be the 1-symmetric edge at v. Let $\{v, w\}$ be the 2-symmetric edge at v that F contains. If $-F$ is contained $\{v, w\}$, then F and $-F$ would have two edges in a row in common, which would cause the vertex-figure at v in $\mathcal{P}$ to be non-polygonal. So $-F$ contains a different 2-symmetric edge at v, and since $\mathcal{P}$ is 3-valent there is a unique other choice $\{v, w'\}$. Recall that F and $-F$ share all of their vertices, and so w' is a vertex of F as well. It follows that if we start at any vertex of F and follow any edge out of it, then we reach another vertex of F, so by the connectivity of $\mathcal{P}$ it follows that F (and $-F$) contain all the vertices of $\mathcal{P}$. If F is a $2k$-gon, then $\mathcal{P}$ has $2k$ vertices, and the dihedral stabilizer of $\mathcal{P}$ acts vertex-transitively on them. However, no dihedral subgroup of $[3, 3]$, $[3, 4]$ or $[3, 5]$ has an orbit of size 12, 24 or 60, respectively, and those are the required numbers of vertices of 3-valent 3-orbit polyhedra for each group. We may conclude that the 2-symmetric faces of $\mathcal{P}^{\zeta_2}$ are polygons, and $\mathcal{P}^{\zeta_2}$ itself is a polyhedron. □

Lemma 4.12. *Let $\mathcal{P}$ be a vertex-transitive 3-orbit polyhedron in class $3^{1,2}$ with planar 1-symmetric faces in general position such that $G(\mathcal{P}) \in \{[3, 3], [3, 4], [3, 5]\}$, and let e be an edge. Then there exists a plane reflection $T \in G(\mathcal{P})$ that preserves e while swapping its ends.*

Proof. First suppose that e is 1-symmetric. Then by Lemma 4.5 there exists $\mathcal{Q}$ vt-equivalent to $\mathcal{P}$ whose 1-symmetric edges do not contain the center of $\mathcal{Q}$. In that situation the three non-trivial symmetries preserving an edge of $\mathcal{Q}$ must be two plane reflections and a half-turn. Since $G(\mathcal{P}) = G(\mathcal{Q})$ and their symmetries act in the same way on their edges, we conclude that e is invariant under two plane reflections; one fixes it pointwise while the other swaps its endpoints.

Now, if e is 2-symmetric let f be the 1-symmetric face that contains it. Then there exists $T \in G(\mathcal{P})$ that preserves f while interchanging the endpoints of e. Since T must fix the axis of the rotation or rotatory reflection that preserves f acting regularly on its edges, and T also must fix the midpoint of e, it must be a plane reflection. □

Now we are ready for the remaining two main theorems of this section.

Theorem 4.13. *Let $\mathcal{P}$ be a vertex-transitive 3-orbit polyhedron in class $3^{1,2}$ in general position with 3-valent vertices and planar 1-symmetric faces such that $G(\mathcal{P}) \in \{[3,3],[3,4],[3,5]\}$. Then $\mathcal{P}$ is vt-equivalent to a truncation of a regular polyhedron.*

Proof. Up to vt-equivalence, we may assume that the convex hulls of the 1-symmetric faces are small, and in particular that they do not intersect. Let f_0 be a 1-symmetric face of $\mathcal{P}$. Recall that its edges are 2-symmetric.

Since f_0 is 1-symmetric, for each of its vertices v there exists $T_v \in G(\mathcal{P})$ that preserves f_0 and v, interchanging the two 2-symmetric edges incident to v. Since v is 3-valent, this means that T_v fixes pointwise the 1-symmetric edge incident to v. From Proposition 4.3 (b) we conclude that T_v is a plane reflection. Furthermore, if e is an edge of f_0 containing v, we know from Lemma 4.12 that the symmetry T_e of $\mathcal{P}$ fixing e while swapping its endpoints is also a plane reflection. We conclude that the stabilizer of f_0 in $G(\mathcal{P})$ is generated by the two plane reflections T_v and T_e. As a consequence, there is a rotation T_eT_v with axis ℓ_0 mapping every edge of f_0 to the next one. This rotation cyclically permutes the 1-symmetric edges incident to the vertices of f_0. See Figure 7.

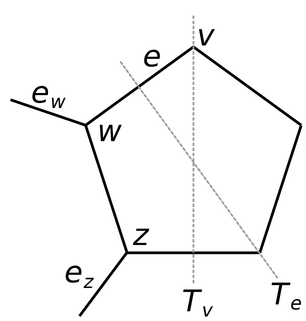


Figure 7: Local picture of the 1-symmetric face f_0

Given any vertex w of f_0 we intend to extend the 1-symmetric edge e_w at w until it meets with ℓ_0. To justify that this can be assumed to occur, note first that both e_w and

ℓ_0 are contained on the invariant plane of the reflection T_w that fixes w while interchanging its two 2-symmetric edges. Now, if ℓ_0 and e_w were parallel then our assumption of the small size of the convex hulls of the 1-symmetric faces implies that the 1-symmetric edges at vertices of f_0 would join f_0 with the 1-symmetric face that winds around ℓ_0 but at the other side of the center of $\mathcal{P}$. If $G(\mathcal{P}) = [3,3]$ there is no such face (the group does not act transitively on the rays of the 3-fold rotation axes), while if $G(\mathcal{P}) \in \{[3,4],[3,5]\}$ then the two faces winding around ℓ_0 and their 1-symmetric edges joining them would form a connected component of the 1-skeleton of $\mathcal{P}$, that is not invariant under the entire group, a contradiction; we conclude that ℓ_0 intersects the line spanned by e_w. Finally, if we are in the situation where ℓ_0 and e_w intersect then we may use vt-equivalence and move w by choosing it in the plane of the reflection T_w as its image under the reflection by ℓ_0 (in this plane), and adjust the remaining vertices accordingly. (See Figure 8.) In that situation the 1-symmetric edges will not intersect the rotation axes preserving 1-symmetric edges. Therefore, it is possible to extend each 1-symmetric edge at both ends until it meets the rotation axes of the 1-symmetric faces containing its vertices. Let S be that graph.

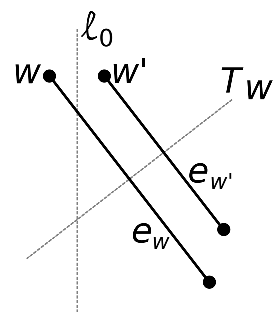


Figure 8: Adjusting the placement of w so that ℓ_0 and e_w do not intersect

Each 2-symmetric face f of $\mathcal{P}$ alternates 1- and 2-symmetric edges. Each 1-symmetric edge e of f has its representative e' in the graph S; furthermore, two consecutive 1-symmetric edges e_1 and e_2 around f have endpoints in a 2-symmetric edge of $\mathcal{P}$, implying that e_1' and e_2' have an endpoint in common (namely, the common intersection of the affine spans of e_1 and e_2 with ℓ_0). It follows that f induces a polygon f' in S. It is easy to see that the union of the polygons f' in S arising from 2-symmetric faces f of $\mathcal{P}$ conform a polyhedron $\mathcal{Q}$. The symmetries of $\mathcal{P}$ all preserve $\mathcal{Q}$, and since the former has 3 times as many edges as the latter, we can conclude that $\mathcal{Q}$ is regular. Clearly, $\mathcal{P}$ is a truncation of $\mathcal{Q}$. □

Theorem 4.14. *Let $\mathcal{P}$ be a vertex-transitive 3-orbit polyhedron in class $3^{1,2}$ in general position with 3-valent vertices and non-planar 1-symmetric faces such that $G(\mathcal{P}) \in \{[3,3],[3,4],[3,5]\}$. Then $\mathcal{P}$ is vt-equivalent to $Tr(\mathcal{Q})^{\varsigma_2}$ for some finite regular polyhedron $\mathcal{Q}$.*

Proof. Proposition 4.11 tells us that $\mathcal{P}^{\varsigma_2}$ has planar 1-symmetric faces, and we may choose the base vertex so that the convex hulls of the 1-symmetric faces do not intersect. By Proposition 4.11, $\mathcal{P}^{\varsigma_2}$ is a polyhedron. Since the 1-symmetric faces of $\mathcal{P}$ are not planar, they must split into two planar 1-symmetric faces of $\mathcal{P}^{\zeta_2}$, none of which contains the center of $\mathcal{P}$ in their convex hull. We can conclude that $\mathcal{P}^{\zeta_2}$ is in general position. An application of Theorem 4.13 concludes the proof. □

Combining Theorems 4.13 and 4.14 with Corollary 4.7 and Proposition 4.11 yields the following corollary:

Corollary 4.15. *The finite vertex-transitive* 3*-orbit polyhedra with* 3*-valent vertices and irreducible symmetry group consist of:*

(a) *The truncations of the finite regular polyhedra,*

(b) *The polyhedra obtained by applying* ζ_2 *to these truncations, and*

(c) *The Petrials of all of the above.*

In particular, there are 72 *such polyhedra.*

According to [11] there are precisely 18 finite regular polyhedra in $\mathbb{E}^3$. In the notation of [5] they are the 5 platonic solids $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$ and the 4 Kepler-Poinsot polyhedra $\{5/2,3\}$, $\{5/2,5\}$, $\{3,5/2\}$, $\{5,5/2\}$ together with their Petrials. From Theorem 4.13 we know that there are precisely 18 polyhedra in class $3^{1,2}$ in general position with planar 1-symmetric faces and irreducible symmetry group: the truncations of the 18 finite regular polyhedra in $\mathbb{E}^2$.

The truncations of the Platonic solids are Archimedean solids and are well-understood. The truncations of $\{3,5/2\}$, $\{5,5/2\}$ have planar faces that can be made regular for a certain choice of base vertex, and therefore are listed in [6] with the numbers 71 and 47, respectively. This is not the case for the truncations of $\{5/2,3\}$ and $\{5/2,5\}$, where the 2-symmetric faces are truncated pentagrams and are not regular for any choice of base vertex. The 2-symmetric faces of the Petrials of the Platonic solids and Kepler-Poinsot polyhedra are truncated skew polygons and therefore are non-planar. The truncations of these 18 polyhedra are all described in [28, Chapter 3] as the polyhedra $P^{0,1}$ listed in each subsection.

Applying ζ_2 to each of the truncations transforms planar 1-symmetric faces into skew 1-symmetric faces and vice-versa, as long as we started with a truncation in general position. Furthermore, in all cases it returns a polyhedron. If $\mathcal{P}$ is a Platonic or Kepler-Poinsot solid then the 2-symmetric faces of $\mathrm{Tr}(\mathcal{P})$ are planar and the vertices of $\mathrm{Tr}(\mathcal{P})^{\zeta_2}$ lie on two parallel planes; each 1-symmetric edge is contained in one of these planes while 2-symmetric edges have one endpoint in each plane. The 2-symmetric faces of the outcome of applying ζ_2 to the truncations of the Petrials of the Platonic and Kepler-Poinsot solids are also non-planar with their vertices lying in two planes, but here every edge has one vertex in each of the planes.

Taking truncation to each of the finite regular polyhedra preserves the symmetry group. This is also the case when performing the ζ_2 operation, with the following exception. If $\mathcal{P}$ is the tetrahedron or its Petrial then $\mathrm{Tr}(\mathcal{P})$ is not centrally symmetric. Therefore $\mathrm{Tr}(\mathcal{P})^{\zeta_2}$

has twice as many vertices; furthermore, its symmetry group acquires the central inversion and it becomes $[3,4]$. For all other choices of finite regular polyhedron $\mathcal{P}$ we have that the central inversion belongs to $G(\mathcal{P})$, and the replacement of all 2-symmetric edges $\{u,v\}$ and $\{-u,-v\}$ of $\mathrm{Tr}(\mathcal{P})$ for $\{u,-v\}$ and $\{-u,v\}$ yields a connected graph. It follows that $G(\mathcal{P}) = G(\mathrm{Tr}(\mathcal{P})^{\varsigma_2})$.

Some of the vt-equivalent realizations of the vertex-transitive 3-orbit polyhedra with 3-valent vertices and symmetry group $[3,3]$ will be provided. Given the current description we shall omit the ones with symmetry groups $[3,4]$ and $[3,5]$.

The vertex-transitive polyhedra with 3-valent vertices and symmetry group $[3,3]$, $[3,4]$, or $[3,5]$ are listed in Tables 3 (in the first four rows), 6 (in the first five groups), 9, and 10.

Finally, let us note that though the polyhedra in Tables 4 and 7 are not geometric truncations, they are combinatorially equivalent to truncated regular maps. Indeed, this follows from [20, Thm. 5.1].

5 Full tetrahedral group

Information about $[3,3]$	
Description	Symmetry group of a tetrahedron $\mathcal{T}$
Order	24
Admissible vertex orbits	4, 6, 12
Involutions	6 plane reflections 3 half-turns whose mirrors join midpoints of edges of $\mathcal{T}$

There are two kinds of subgroups of $[3,3]$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$: the group consisting of the three half-turns and the identity, and the group generated by two reflections with perpendicular mirrors. Recall that for all classes of 3-orbit polyhedra with full tetrahedral symmetry group, there is an orbit of edges whose edge stabilizer is of this kind.

5.1 Vertex-intransitive

Vertex-intransitive polyhedra with group $[3,3]$	
1-symmetric edges	6
2-symmetric edges	12
1-symmetric vertices	four 3-valent
2-symmetric vertices	four 6-valent or six 4-valent

We first consider 3-orbit polyhedra in class $3^{0,1}$. Let $\mathcal{P}$ be one such polyhedron.

We first discard the possibility of $\mathcal{P}$ having 6 vertices that are 2-symmetric. Recall that the points whose orbit under $[3,3]$ have 6 elements are in the axes of the half-turns (aligned with the midpoints of edges of $\mathcal{T}$). An edge e between two of these vertices must be 1-symmetric, and such an edge is invariant under a subgroup of $[3,3]$ with 4 elements. However, each reflection plane of $[3,3]$ contains only one axis of a half-turn in $[3,3]$, which implies that the endpoints of e must belong to the same axis of a half-turn. This would imply that there is only one 1-symmetric edge incident to a 2-symmetric vertex, contradicting Lemma 2.1.

Now we know that all 3-orbit polyhedra in class $3^{0,1}$ with full tetrahedral symmetry group have four 1-symmetric vertices, and four 2-symmetric vertices. The four vertices in each orbit must lie on the axes of the 3-fold rotations (each determined by a vertex and the center of the opposite triangle in $\mathcal{T}$).

Since the 2-symmetric vertices have degree 6, each such vertex v is adjacent to the other three 2-symmetric vertices and to the three 1-symmetric vertices that are not in the same axis as v. Up to vi-equivalence, we may assume that all vertices lie on a sphere, where vertices in the same rotation axis are at opposite sides of the center of $\mathcal{P}$. In this way we get a graph vi-equivalet to the 1-skeleton of the convex Catalan solid called triakis tetrahedron (see Figure 9), the dual of the truncated tetrahedron.

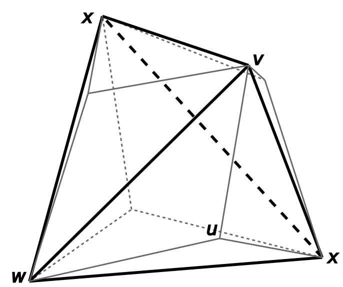


Figure 9: Triakis tetrahedron; thick black edges belong to the underlying tetrahedron

In the 1-skeleton of the triakis tetrahedron the only possible 3-symmetric faces are precisely the triangles of the triakis tetrahedron. If we assume otherwise then we must skip either 2 or 4 edges at the 2-symmetric vertices. If u is a 1-symmetric vertex and its neighbors in a face are v and w, by skipping 2 or 4 edges at v and w we reach a vertex x adjacent by 1-symmetric edges in the same face both to v and w, preventing the face to be a polygon, or to be be 3-symmetric (see Figure 9, where the two possibilities of x are shown). We can conclude the following result.

Proposition 5.1. *Up to similarity and vi-equivalence the triakis tetrahedron is only one 3-orbit polyhedron in class $3^{0,1}$ in $\mathbb{E}^3$.*

5.2 Vertex-transitive cases

Vertex-transitive polyhedra with group $[3,3]$	
1-symmetric edges	6
2-symmetric edges	12
Vertices	twelve 3-valent

The only possibility is to have twelve 3-valent vertices, and so the results in Subsection 4.3 imply the following.

Proposition 5.2. *Up to similarity and vt-equivalence the only vertex-transitive* 3*-orbit polyhedra in* $\mathbb{E}^3$ *with symmetry group* $[3,3]$ *are:*

- *the truncated tetrahedron* $\mathrm{Tr}(\{3,3\})$ *in class* $3^{1,2}$*,*
- *the truncated hemicube* $\mathrm{Tr}(\{4,3\}_3)$ *in class* $3^{1,2}$*,*
- $\mathrm{Tr}(\{3,3\})^\pi$ *in class* 3^1*,*
- $\mathrm{Tr}(\{4,3\}_3)^\pi$ *in class* 3^1*.*

Proof. We know from Theorems 4.13 and 4.14 that every polyhedron in class $3^{1,2}$ is either the truncation of a regular polyhedron, or the image under ζ_2 of that truncation. The only regular polyhedra with symmetry group $[3,3]$ are the tetrahedron and the hemi-cube, yielding the first two items in the statement. However, as noted in Subsection 4.3, when applying ζ_2 to either polyhedra in the first two items the symmetry group increases to $[3,4]$. We conclude that the only polyhedra in class $3^{1,2}$ are $\mathrm{Tr}(\{3,3\})$ and $\mathrm{Tr}(\{4,3\}_3)$. Corollary 4.7 is used directly to find all polyhedra in class 3^1 from those in class $3^{1,2}$. □

To conclude this section we illustrate the two essentially different kinds of realizations of the 1-skeleton of the truncated tetrahedron (and hence of each of the vertex-transitive 3-orbit polyhedra with symmetry group $[3,3]$). For sake of brevity, we shall not do the same for the groups $[3,4]$ or $[3,5]$.

The base vertex v_0 must belong to the mirror Π of some plane reflection that fixes pointwise the 1-symmetric edge e_0 containing v_0. We may choose the base vertex as an interior point of the edge of $\mathcal{T}$ or of the edge of the dual of $\mathcal{T}$ contained in Π; in these cases the convex hull is a truncated tetrahedron (possibly with non-regular hexagons) as in Figure 10 (e). Alternatively, v_0 may be chosen as an interior point in the altitude of a triangle of $\mathcal{T}$ contained in Π, that does not project to an edge of the dual of $\mathcal{T}$ from the center of $\mathcal{T}$; in this situation the convex hull is a cuboctahedron (possibly with rectangles instead of squares, and with two sets of triangles of distinct sizes) as in Figure 10 (a). In both figures $\mathcal{T}$ is shown in gray lines.

The 1-symmetric edges are completely determined; one such edge e is illustrated in Figure 10 (b) and (f) for each case of vertex-set. The remaining 1-symmetric edges are the images of e of under $[3,3]$; each of them is parallel to an edge of $\mathcal{T}$. There are two possible choices of 2-symmetric edges depending on the choice of the symmetry that swaps their endpoints. They must be plane reflections, and up to conjugacy by elements in $G(\mathcal{P})$ they can be chosen between a pair of perpendicular reflection plains of $[3,3]$. A triangle for the two distinct choices is shown in Figure 10 (c), (d), (g), (h).

We summarize the enumeration in this section with the following theorem.

Theorem 5.3. *There are* 5 *vertex-transitive* 3*-orbit polyhedra with symmetry group* $[3,3]$*, summarized in Table 3.*

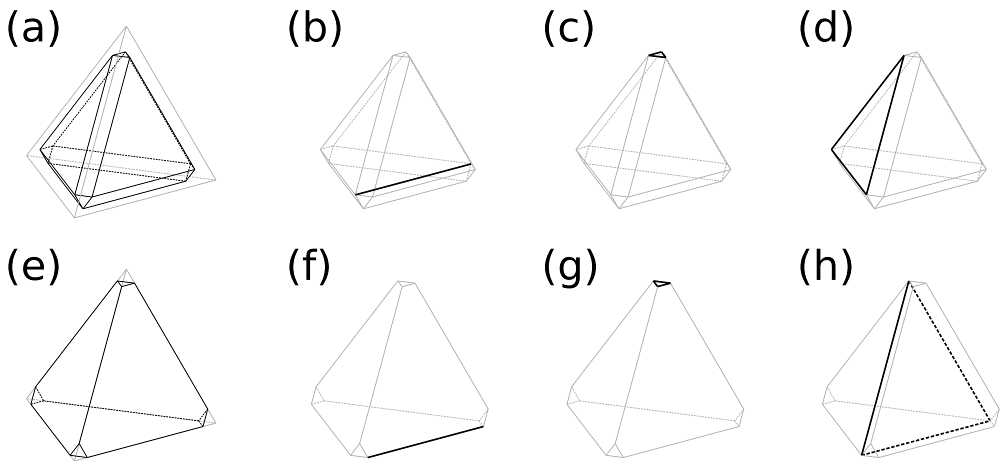


Figure 10: Edges of the vertex-transitive 3-orbit polyhedra

6 Rotational octahedral group

Information about $[3,4]^+$	
Description	Orientation-preserving symmetries of a cube $\mathcal{C}$
Order	24
Admissible vertex orbits	6, 12
Involutions	3 half-turns with mirrors parallel to the edges of $\mathcal{C}$ 6 half-turns with mirrors joining midpoints of opposite edges of $\mathcal{C}$.

Theorem 4.2 proves that $\mathcal{P}$ cannot be in class $3^{0,1}$, and so we will assume that $\mathcal{P}$ is vertex-transitive.

Vertex-transitive polyhedra with group $[3,4]^+$	
1-symmetric edges	6
2-symmetric edges	12
Vertices	Twelve 3-valent vertices

The vertex set of $\mathcal{P}$ consists of the midpoints of the edges of $\mathcal{C}$. The convex hull of the vertex set is an Archimedean cuboctahedron. The 1-symmetric edges of $\mathcal{P}$ must be those joining two antipodal vertices (midpoints of edges of $\mathcal{C}$), since they are the only segments between pairs of vertices that are fixed pointwise by non-trivial involutions in $[3,4]^+$.

To determine the 2-symmetric edges let v_0 be a vertex of $\mathcal{P}$ and e_0 be the 1-symmetric edge joining v_0 to its antipode. A 2-symmetric edge e_1 at v_0 must be invariant under an involution $T_1 \in [3,4]^+$ that swaps its endpoints. We claim that this involution cannot be a half-turn about the axis of 4-fold rotations. Indeed, those half-turns only preserve segments between midpoints of antipodal edges of $\mathcal{C}$ and altitudes of squares of $\mathcal{C}$, and in each of those cases the 1-skeleton of $\mathcal{P}$ would be disconnected.

Among the six axes of half-turns of $[3,4]^+$ that are not axes of 4-fold rotations, one fixes pointwise e_0, one fixes e_0 while interchanging its endpoints, and the other four are equivalent under the stabilizer of v_0 in $[3,4]$, and therefore yield isometric structures (in left- and right-handed versions). Figure 11 (a) illustrates a vertex (black node) with its 1-symmetric edge (dashed line) and the four choices of 2-symmetric edges at that vertex.

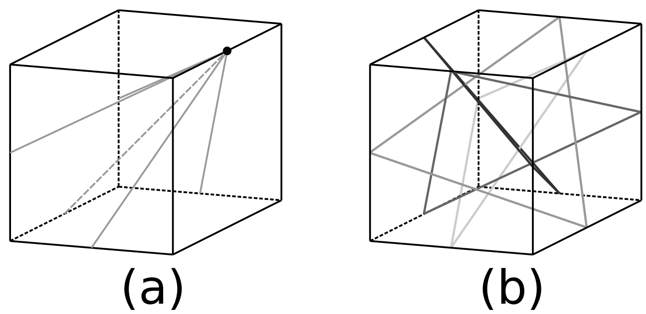


Figure 11: Edges and 1-symmetric faces of the vertex-transitive 3-orbit polyhedra

The orbit under $[3,4]^+$ of one of the candidates of 2-symmetric edges induces four triangles with disjoint vertex sets. The vertices of any of these triangles are the midpoints of three disjoint edges of a Petrie path of $\mathcal{C}$, and hence the center of every triangle is the center of $\mathcal{C}$. An orbit of 2-symmetric edges is illustrated in Figure 11 (b), where the triangles are in distinct shades of gray.

The 1-skeleton of $\mathcal{P}$ is a connected, vertex-transitive cubic graph with girth 3 on 12 vertices. According to [22], it is isomorphic to the 1-skeleton of the truncated tetrahedron (the only graph with those properties). The faces of $\mathcal{P}$ are the only possible ones in the 1-skeleton of the truncted tetrahedron, described in Proposition 5.2, with the only difference being the way of realizing the truncated tetrahedron. We will denote this 1-skeleton by $H_{[4,3]}$ and think of it as a realization of the 1-skeleton of the truncated hemi-cube; we will see later that the polyhedra with symmetry group $[3,5]^+$ can similarly be described in terms of the truncated hemi-dodecahedron and truncated hemi-icosahedron.

Theorem 6.1. *There are four 3-orbit polyhedra in $\mathbb{E}^3$ with $G(\mathcal{P}) = [3,4]^+$, summarized in Table 4. In each case, the convex hull of the vertex set is an Archimedean cuboctahedron.*

To reinforce the visualization of the faces of the polyhedra in Theorem 6.1, Figure 13 shows an Archimedean cuboctahedron with its vertices labeled. Vertices 1, 2, 3, 4 correspond to the upper square while vertices 9, 10, 11, 12 correspond to the lower square. A sample 9-gon and a sample 12-gon of polyhedra in class 3^1 with symmetry group $[3,4]^+$ are given by the lists of vertices $(2,7,5,11,4,10,8,3,9)$ and $(2,7,5,4,11,1,12,6,8,10,3,9)$, respectively. A 2-symmetric 6-gon and a 2-symmetric 8-gon of polyhedra in class $3^{1,2}$ with symmetry group $[3,4]^+$ are given by the lists of vertices $(5,4,10,3,9,7)$ and $(5,4,10,8,6,12,2,7)$, respectively.

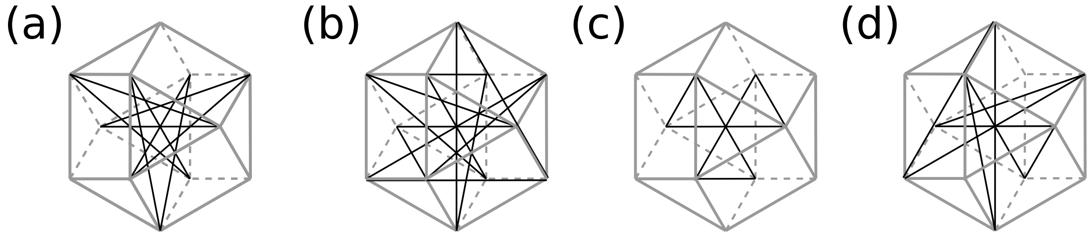


Figure 12: 3-symmetric and 2-symmetric faces of 3-orbit polyhedra with symmetry group $[3, 4]^+$

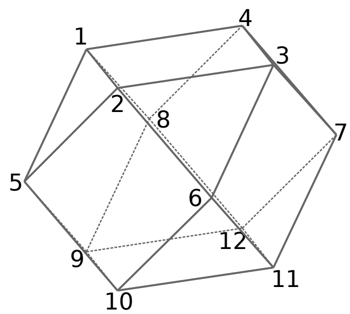


Figure 13: Archimedean cuboctahedron

These two last polygons can be constructed respectively from opposite triangles and opposite squares of the cuboctahedron.

7 Full octahedral group

Information about $[3, 4]$	
Description	Symmetries of a cube $\mathcal{C}$
Order	48
Admissible vertex orbits	6, 8, 12, 24
Involutions	Central inversion 3 half-turns with mirrors parallel to the edges of $\mathcal{C}$ 6 half-turns with mirrors joining midpoints of opposite edges of $\mathcal{C}$ 3 reflections in mirrors that are parallel to the faces of a cube 6 reflections in mirrors that contain pairs of opposite edges of a cube

7.1 Class $3^{0,1}$

Vertex-intransitive polyhedra with group $[3,4]$	
1-symmetric edges	12
2-symmetric edges	24
1-symmetric vertices	six 4-valent or eight 3-valent
2-symmetric vertices	six 8-valent, eight 6-valent, or twelve 4-valent

We start by determining the graph induced by the 1-symmetric edges between 2-symmetric vertices.

If there are six 2-symmetric vertices, then each one is incident to four 2-symmetric vertices. Thus the 1-symmetric edges must form the 1-skeleton of an octahedron. If there are eight 2-symmetric vertices, each incident to three others, then the 1-symmetric edges either form the graph of a cube or of two disjoint tetrahedra inscribed in a cube. Finally, if there are twelve 2-symmetric vertices, each incident to two others, then the 1-symmetric edges must belong to mirrors of plane reflections and therefore form three disjoint 4-cycles. We split our analysis according to the three possibilities of graph induced by the 2-symmetric vertices.

7.1.1 Octahedral graph

Our first case is when the 2-symmetric vertices induce the 1-skeleton of an octahedron. We split into cases depending on the number of 1-symmetric vertices.

Six 1-symmetric vertices

Each 1-symmetric vertex is incident to four 2-symmetric vertices. Up to vi-equivalence, we may consider these vertices as the centers of faces of $2\mathcal{C}$, and the stabilizer of each 1-symmetric vertex has a single orbit of 2-symmetric vertices of size four, giving us the graph of a cloned octahedron. We will denote the 2-symmetric vertices as 1 through 6 as in Figure 14, and the 1-symmetric vertices will be denoted $1'$ through $6'$, with (for example) $1'$ being adjacent to the neighbors of 1.

Now, let F be a face containing the 1-symmetric edge $(2,3)$. The nontrivial symmetry of $\mathcal{P}$ that fixes both endpoints is a reflection in a plane through vertices 2, 3, 4, and 5. Furthermore, this plane also contains vertices $2'$, $3'$, $4'$, and $5'$. By Lemma 2.5, a face that starts with $(2,3)$ cannot continue to $2'$ or $4'$. Then we may assume that F starts with $(2,3,1')$.

There are two symmetries that fix that edge while interchanging its endpoints: A reflection through the bisector of that edge, and a half-turn about a line through the center of that edge. One of these two symmetries must fix F. If the reflection fixes F, then F must contain the edge from $1'$ to 2 in addition to the edge from $1'$ to 3, and so we get triangular faces. The vertex-figure at 2 consists of two disjoint 4-cycles; the triangles $231', 251', 256', 236'$ induce one of the 4-cycles. So we do not get a polyhedron this way. Thus, we may assume that a half-turn about the center of $(2,3)$ fixes F, and so F starts with $(6',2,3,1')$. The image of

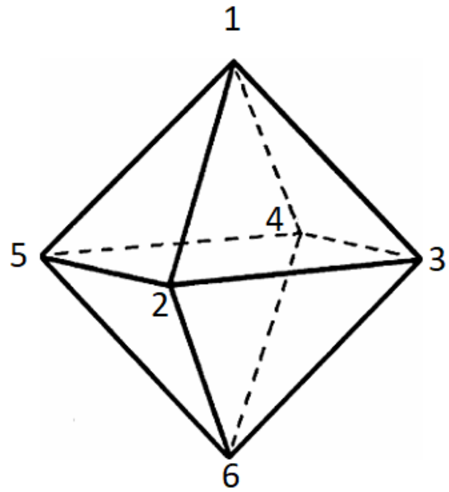


Figure 14: Labeling of vertices of the octahedron

this partial face under the symmetry group of $\mathcal{P}$ includes a face that starts $(6', 2, 5)$, one that starts $(1', 2, 5)$, and one that starts $(1', 2, 3)$. So again, we get a 4-cycle in the vertex-figure at 2, and so there are no polyhedra in this case.

Eight 1-symmetric vertices

Up to vi-equivalence, we may consider the 1-symmetric vertices to be the vertices of the cube $0.5\mathcal{C}$. Each 1-symmetric vertex is incident to three 2-symmetric vertices. There are two choices: we can connect each 1-symmetric vertex to the nearest three 2-symmetric vertices or the furthest three. But these are vi-equivalent; switching every 1-symmetric vertex v with its opposite $-v$ induces the equivalence. Thus we will assume that the 1-symmetric vertices are incident to the closest three 2-symmetric vertices. This gives us the 1-skeleton of a *triakis octahedron*.

We keep the labeling of the 2-symmetric vertices as in Figure 14, and label the 1-symmetric vertices as in Figure 15. In this way, 2-symmetric vertex 4 can be thought of as the center of the square determined by the 1-symmetric vertices c, d, h, g, and hence we assume that there is an edge from 4 to each of c, d, h and g. On the other hand, the edges in Figure 15 are only a reference; they are not edges of the triakis octahedron.

Let F be a face containing the edge $(2, 3)$. Up to symmetry, we may assume that the face contains either $(2, 3, b)$ or $(2, 3, c)$. Suppose that the reflection through the middle of that edge fixes F. If F contains $(2, 3, b)$, then this reflection gives us the edge between 2 and b, and so we get triangular faces (see Figure 16 (a)). This gives us the triakis octahedron. If instead, F contains $(2, 3, c)$, then it also contains the edge $(2, a)$. Lemma 2.3 implies that the vertices 1 and 6 cannot be part of F. Considering the orbit of the partial face $(a, 2, 3, c)$ under the stabilizer of a, and throwing away those pieces that contain 1, 6, or 2 (since we have already used 2), we see that $(a, 5, 4, c)$ must also be part of F. Thus, the faces are

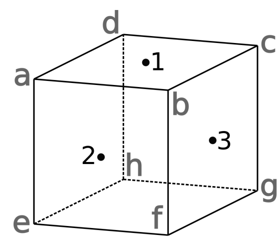


Figure 15: Labeling of 1-symmetric vertices

hexagons in the orbit of $(a, 2, 3, c, 4, 5)$ (see Figure 16 (b)). It is straightforward to verify that the vertex-figures are all connected, and so this yields a polyhedron.

Finally, suppose that the half-turn through the middle of $(2, 3)$ fixes F. Note that this half-turn does not fix any vertex, and so Remark 2.9 implies that F has an even number of vertices. Since 2/3 of the vertices of F must be 2-symmetric, that implies that F is a hexagon. Suppose F contains $(2, 3, b)$. Then it also contains the edge $(2, f)$. Then, since b is only adjacent to vertices 1, 2, and 3, and 2 is already incident to two vertices of F, it follows that the edge $(b, 1)$ is in F, and then applying the half-turn we see that $(f, 6)$ is in F. So F must contain $(6, f, 2, 3, b, 1)$. Since F is a hexagon, this would have to be the whole face – however, 1 is not adjacent to 6. So the last possibility is for F to contain $(2, 3, c)$; applying the half-turn symmetry we get that F contains $(e, 2, 3, c)$. Now, if F contains $(c, 1)$ then it also contains $(e, 6)$, and we would get that $F = (6, e, 2, 3, c, 1)$, which again doesn't work since 1 and 6 are not adjacent. The only remaining possibility is $F = (5, e, 2, 3, c, 4)$ (see Figure 16 (c)). This is a hexagon, the vertex-figures are connected, and so we get a polyhedron. Note that the two polyhedra with hexagonal faces are not combinatorially equivalent; in the first case, each 2-symmetric vertex that is not contained in a given face F is adjacent to either 0 or 2 1-symmetric vertices of F, whereas in the second case, each such 2-symmetric vertex is adjacent to one 1-symmetric vertex of F.

We have proven the following result.

Proposition 7.1. *Up to similarity and vi-equivalence there are three* 3*-orbit polyhedra* $\mathcal{P}$ *with* $G(\mathcal{P}) = [3, 4]$ *and with six* 2*-symmetric vertices. Their* 1*-skeleton is the triakis octahedron. One face of each of them is illustrated in Figure 16.*

7.1.2 Eight 2-symmetric vertices

Now suppose that there are eight 2-symmetric vertices, each of degree 6. These vertices can be understood as the vertices of $\mathcal{C}$. Each vertex must be incident to three other vertices of this type, and so the 1-symmetric edges (which connect 2-symmetric vertices) must either form the skeleton of a cube, or else that of the stella octangula, that is, two disjoint 1-skeleta of tetrahedra inscribed in the cube. Again we split our analysis according on the number (6 or 8) of 1-symmetric vertices.

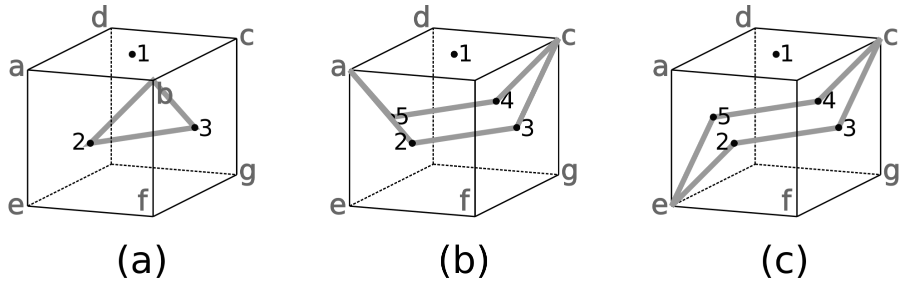


Figure 16: Faces of 3-orbit polyhedra in class $3^{0,1}$ with six 2-symmetric vertices

Six 1-symmetric vertices

Each 1-symmetric vertex is incident to four 2-symmetric vertices. Up to vi-equivalence, we may consider these vertices as the centers of faces of $2\mathcal{C}$, each connected to the vertices of the corresponding face of the cube. If the graph of 1-symmetric edges is a cube, then the full graph is the 1-skeleton of the tetrakis hexahedron. Otherwise, it can be thought as the 1-skeleton of the stella octangula where 6 vertices are added at the intersections of the edges, together with 24 edges from these 6 vertices along halves of the original edges of the stella octangula. We will denote the 2-symmetric vertices by 1 through 8, and the 2-symmetric vertices will be a through f (see Figure 17), where d, e and f are opposite to b, c and a, respectively.

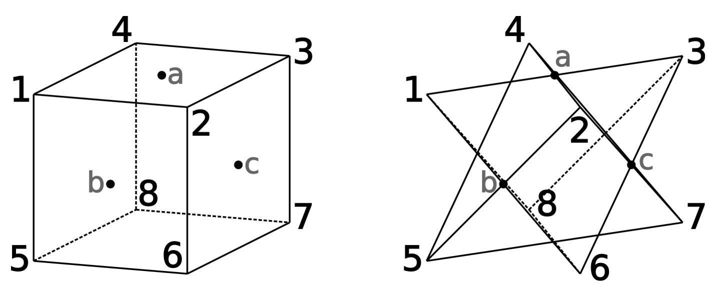


Figure 17: Labeling of vertices

Let us start by assuming that the graph of 1-symmetric edges is a cube. Consider the 1-symmetric edge $(1, 2)$, and let F be a face containing this edge. By Lemma 2.5, the next vertex cannot be c. Then without loss of generality, the face contains $(1, 2, a)$. If the reflection through the bisector of $(1, 2)$ fixes F, then that implies that F is the triangle $(1, 2, a)$. In this case, we get the tetrakis hexahedron. Now suppose that the half-turn at $(1, 2)$ fixes F. Note that this symmetry does not fix any vertices, and so F must be either a hexagon or a 12-gon. Now, applying the half-turn to $(1, 2, a)$ we see that F contains $(b, 1, 2, a)$. In order for F to be a hexagon, there would need to be adjacent numbered vertices, different from 1 and 2, such that one of them is adjacent to a and the other to b. There is no such pair of vertices, and so F must be a 12-gon. There are two possibilities: one where the faces “pass through” the lettered vertices as $(b, 1, 2, a, 4, 3, d, 8, 7, f, 5, 6)$, and one where the faces "turn" at the lettered vertices as $(b, 1, 2, a, 3, 4, d, 8, 7, f, 6, 5)$. But "passing through" gives a disconnected vertex-figure at a. The other choice gives us 12-gonal faces, while the vertex-figures at the 1-symmetric vertices are squares and those at the 2-symmetric vertices are hexagons.

Now consider the graph represented on the right in Figure 17, and the 1-symmetric edge $(1, 3)$. Let F be a face containing this edge. By Lemma 2.5, the next vertex cannot be a. Without loss of generality, F contains $(1, 3, c)$. Suppose the reflection T through the bisector of $(1, 3)$ fixes F. This reflection fixes vertices 2, 4, 6, and 8, and so Lemma 2.3 implies that none of these vertices are part of F. Now, applying T to $(1, 3, c)$ shows us that F contains $(b, 1, 3, c)$, and then the only possible face is $(b, 1, 3, c, 7, 5)$. This yields connected vertex-figures at every vertex and we get a polyhedron. Indeed, the vertex-figures at 1-symmetric vertices are squares, whereas those at 2-symmetric vertices are prismatic skew hexagons. Now, suppose instead that the half-turn T' through the center of $(1, 3)$ fixes F. In order to get connected vertex-figures at c, F must contain $(1, 3, c, 2)$ or $(1, 3, c, 7)$. In the first case, applying T' gives us that F contains $(4, e, 1, 3, c, 2)$. There must be a symmetry of F that sends the arc $(4, e)$ to $(3, c)$; indeed there are two such symmetries, both of order 2: the reflection through the bisector of $(3, 4)$ and the half-turn through a line containing the center of $(3, 4)$. Since these symmetries have order 2, it follows that F is the hexagon $(4, e, 1, 3, c, 2)$, and the symmetry of F mapping $(4, e)$ to $(3, c)$ is in fact the reflection. This gives us connected vertex-figures (the same as in the case when the stabilizer of $(1, 3)$ in $G(F)$ is T instead of T'). The second case, where F contains $(1, 3, c, 7)$, follows similarly and we get a polyhedron with a sample hexagonal face $(5, e, 1, 3, c, 7)$.

Let us show that the three polyhedra just produced are all combinatorially distinct, considering Figure 18. We can distinguish (c) from the others by noting that the two 1-symmetric vertices of a face have two common neighbors in (c) but none in common in Figure 18 (d) and (e). Furthermore, we can distinguish (d) from (e) by noting that the four 2-symmetric vertices of a face are all incident to a single 2-symmetric vertex in (d) but not in (e).

Eight 1-symmetric vertices

Up to vi-equivalence, we may consider the 1-symmetric vertices to be the vertices of the cube $1.2\mathcal{C}$. Each 1-symmetric vertex v is incident to three 2-symmetric vertices. They can be either the ones corresponding to the neighbors of v in $1.2\mathcal{C}$, or the antipodes of the neighbors (corresponding to vertices at distance 2 from v in $1.2\mathcal{C}$).

If the graph induced by the 2-symmetric vertices is a cube, then we use that antipodes belong to the same axis of 3-fold rotation and therefore these two choices yield vi-equivalent polyhedra. The 1-skeleton is isomorphic to the cloned cube. On the other hand, if the graph induced by the 2-symmetric vertices is the union of the two tetrahedra inscribed in $\mathcal{C}$ then the three edges at a given 1-symmetric vertex join it with three 2-symmetric vertices in the same tetrahedron, and therefore the 1-skeleton becomes disconnected (the union of two disjoint triakis tetrahedra). Hence we assume that the 1-skeleton is isomorphic to the cloned cube.

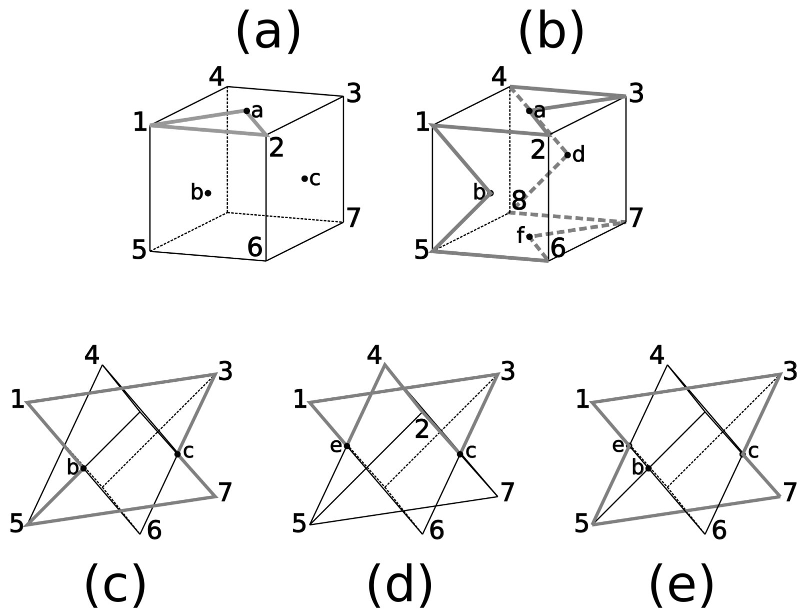


Figure 18: Faces of 3-orbit polyhedra in class $3^{0,1}$ with eight 2-symmetric vertices

Note here that the vertex-figures should all be polygons, as long as the two neighbors of a 2-symmetric vertex on a give face belong to distinct axes of 3-fold rotation. The 1-symmetric vertices are all 3-valent, which implies that the vertex-figures are triangles. On the other hand, the 2-symmetric vertices are all 6-valent, which forces connectivity of the vertex-figures because of the way the different edge orbits are arranged around a 2-symmetric vertex.

We keep the labeling of 2-symmetric vertices as in the left of Figure 17; then the 1-symmetric vertex cloned to vertex j is called j'. Let F be a face containing the edge $(1, 2)$. By Lemma 2.5, since the reflection through the plane containing vertices 1, 2, 7, and 8 also fixes $1'$, the next vertex after 2 cannot be $1'$. Then without loss of generality, F contains $(1, 2, 3')$. If the reflection that interchanges 1 and 2 is a symmetry of F, then F contains $(4', 1, 2, 3')$. Since this reflection does not fix any vertex, Remark 2.9 implies that F has an even number of sides. If F is a 12-gon, then the 1-symmetric edges of F consist of four disjoint edges in the orbit of $(1, 2)$ under a cyclic subgroup of the symmetry group. The only such orbits are induced by either a quarter-turn about the center of the face $(2, 3, 6, 7)$ of the cube, or a 4-fold rotatory reflection. In all cases the image of the partial face $(4', 1, 2, 3')$ under the subgroup is disconnected. So F must be a hexagon. Then there must be a symmetry of F that fixes $3'$ and $4'$ while sending $(1, 2)$ to the other 1-symmetric edge of F; this must be the plane reflection through $3', 4', 5'$, and $6'$, which sends $(1, 2)$ to $(7, 8)$. Thus $F = (4', 1, 2, 3', 7, 8)$.

If the half-turn is what fixes F while interchanging 1 and 2, then again Remark 2.9 implies that F has an even number of sides, and now F contains $(5', 1, 2, 3')$. An argument similar to the above shows that F cannot be a 12-gon. The only hexagon that contains $(5', 1, 2, 3')$ and is stabilized by the half-turn is $(5', 1, 2, 3', 7, 8)$.

To see that the two polyhedra we obtain are combinatorially distinct, consider all of the neighbors of the 1-symmetric vertices of a face. In the first type of face, the remaining two 2-symmetric vertices are adjacent, but in the second type they are not.

The polyhedra found in this case are enumerated in the following proposition.

Proposition 7.2. *Up to similarity and vi-equivalence there are seven* 3*-orbit polyhedra* $\mathcal{P}$ *in class* $3^{0,1}$ *with* $G(\mathcal{P}) = [3,4]$ *with eight* 2*-symmetric vertices. Two of them have the tetrakis hexahedron as their* 1*-skeleton, and three have as* 1*-skeleton the graph represented on the right of Figure 17. Figure 18 shows in gray sample faces of each of these five polyhedra. The remaining two polyhedra have the cloned cube as their* 1*-skeleton and a sample face for each polyhedron is shown in Figure 19, where the edges incident to the* 1*-symmetric vertices are omitted if they are not part of the face.*

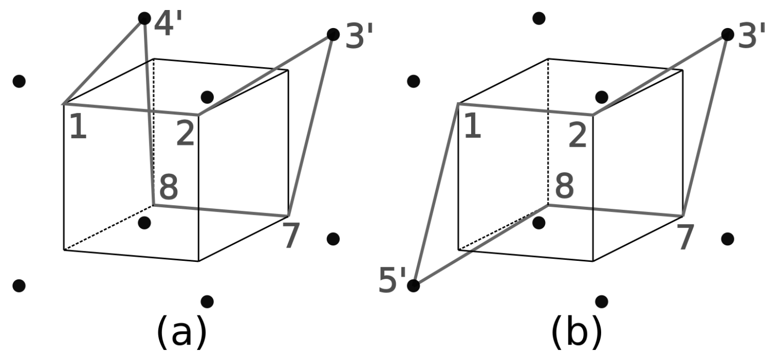


Figure 19: Faces of 3-orbit polyhedra in class $3^{0,1}$ with a cloned cube as 1-skeleton

7.1.3 Twelve 2-symmetric vertices

Finally, suppose that there are twelve 2-symmetric vertices, each of degree 4. These vertices can be understood as the midpoints of edges of $\mathcal{C}$ and we label them as in Figure 20. Each vertex must be incident to two other vertices of this type, and so the 1-symmetric edges (which connect 2-symmetric vertices) form three disjoint 4 cycles. We split the analysis on whether there are six or eight 1-symmetric vertices.

Six 1-symmetric vertices We assume that the 1-symmetric vertices are the centers of the squares of $\mathcal{C}$, labeled as in Figure 20, with vertices 4, 5 and 6 opposite to 2, 3 and 1, respectively. Up to vi-equivalence, there are two graphs: we can either connect each 1-symmetric vertex with the edge midpoints of the same face, or with the midpoints of the edges that are orthogonal to the face.

In the first case, consider the 1-symmetric edge (e, f). The 1-symmetric vertices that are adjacent to f are 2 and 3, both of which are fixed by the plane reflection through e, f, and g. Then Lemma 2.5 implies that this graph is impossible for a 1-skeleton of a 3-orbit polyhedron.

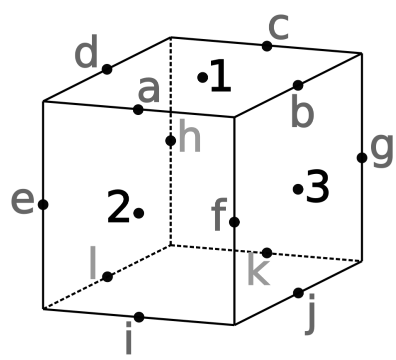


Figure 20: Labels of the twelve 2-symmetric vertices

In the second case, we get a disconnected graph. For example, one of the connected components consists of vertices 1 and 6 along with e, f, g, and h. In any case, we do not get a polyhedron with this vertex set.

Eight 1-symmetric vertices Now the 1-symmetric vertices are assumed to be the vertices of $\mathcal{C}$, labeled as in Figure 17. We must connect the eight 1-symmetric vertices each to three 2-symmetric vertices. Up to vi-equivalence, the only choice is to connect each vertex of the cube to the midpoints of the incident edges, since the remaining 2-symmetric vertices are in an orbit of size 6 under the stabilizer in $[3, 4]$ of the given 1-symmetric vertex. We will denote this 1-skeleton by C.

The 1-symmetric vertices are 3-valent, and hence their vertex-figures are triangles. On the other hand, the vertex-figure of a 2-symmetric vertex is a quadrilateral, since the two edges incident to it on a face do not belong to the same reflection plane of $[3, 4]$ (one edge is 1-symmetric and the other is 2-symmetric). Therefore, any 3-symmetric polygon in this graph is a face of a polyhedron in class $3^{0,1}$.

Without loss of generality, a face F containing (e, f) contains $(e, f, 2)$. Suppose that the symmetry that fixes F and (e, f) while interchanging e and f is the reflection through a, c, and i. Then by Lemma 2.3, F does not contain a, and so the face must continue to b. In this case, the face must be the hexagon $(e, f, 2, b, d, 1)$.

Now, if the symmetry that fixes F and (e, f) while interchanging e and f is a half-turn, then since this half-turn fixes no vertices, Remark 2.9 implies that F has even length. Since F contains $(e, f, 2)$, it must also contain the vertex 5, and then the only way to get a hexagon is with $(e, f, 2, a, i, 5)$. If F is a 12-gon, then its set of 1-symmetric edges must be the orbit under a cyclic subgroup of order 4 of (e, f) that contains four disjoint edges. There is only one such orbit; namely $(e, f), (d, b), (g, h)$, and (j, l). The only possible face that contains these and $(5, e, f, 2)$ is the face $(5, e, f, 2, b, d, 4, h, g, 7, j, l)$. Finally, if F is an 18-gon, then the 1-symmetric edge of F that follows (e, f) must be the image of (e, f) under a rotatory reflection of order 6. In that situation the symmetries stabilizing 1-symmetric vertices while swapping their endpoints should be half-turns, and the conjugation by any such half-turn should invert the rotatory reflection. However, conjugation by a half-turn whose axis contains centers of two opposite squares does not invert any of the four 6-fold rotatory reflections in $[3, 4]$, and hence faces cannot have 18 vertices.

Note that the two polyhedra with hexagonal faces can be distinguished by considering whether the 1-symmetric vertices of a face have a common neighbor or not.

The polyhedra with twelve 2-symmetric vertices are enumerated in the next result.

Proposition 7.3. *Up to similarity and vi-equivalence there are three* 3*-orbit polyhedra* $\mathcal{P}$ *in class* $3^{0,1}$ *with* $\Gamma(\mathcal{P}) = [3, 4]$ *and twelve* 2*-symmetric vertices. The* 1*-skeleton is that of the cube with subdivided edges on their halves, and both altitudes of each square, denoted* C*. A sample face for each polyhedron is shown in Figure 21.*

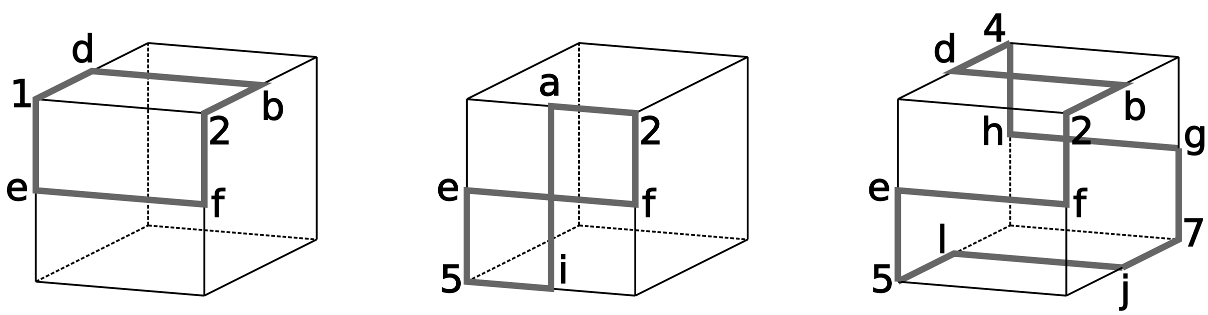


Figure 21: Faces of 3-orbit polyhedra in class $3^{0,1}$ with twelve 2-symmetric vertices

Let us summarize.

Theorem 7.4. *Up to similarity and vi-equivalence, there are thirteen vertex-intransitive* 3*-orbit polyhedra with symmetry group* $[3, 4]$*, summarized in Table 5.*

7.2 Vertex-transitive

Vertex-transitive polyhedra with group $[3, 4]$	
1-symmetric edges	12
2-symmetric edges	24
Vertices	twelve 6-valent or twenty-four 3-valent

If there are 24 vertices, then the stabilizer of each of them is generated by a plane reflection. Applying the results in Subsection 4.3 we obtain the following theorem. Recall that if the symmetry group of $\mathcal{P}$ is $[3, 3]$ then that of $\mathcal{P}^{\zeta_2}$ must be $[3, 4]$.

Proposition 7.5. *Up to similarity and vt-equivalence there are* 20 *vertex-transitive* 3*-valent* 3*-orbit polyhedra in* $\mathbb{E}^3$ *with symmetry group* $[3, 4]$*:*

- *the truncations of the cube* $\{4, 3\}$*, of the octahedron* $\{3, 4\}$*, of* $\{4, 3\}^\pi$ *and of* $\{3, 4\}^\pi$*, all in class* $3^{1,2}$*;*

- *$Tr(\{4,3\})^{\zeta_2}$, $Tr(\{3,4\})^{\zeta_2}$, $Tr(\{4,3\}^{\pi})^{\zeta_2}$, $Tr(\{3,4\}^{\pi})^{\zeta_2}$, all in class $3^{1,2}$;*
- *$Tr(\{3,3\})^{\zeta_2}$, $Tr(\{4,3\}_3)^{\zeta_2}$, both having as 1-skeleton that of $Tr(\{4,3\}^{\zeta})$ and both in class $3^{1,2}$;*
- *the Petrials of each polyhedron in the previous items, all in class 3^1.*

For the remainder of the subsection we assume that there are 12 vertices, each lying on exactly one of the axes of half-turns about edge midpoints. These are the vertices of an Archimedean cuboctahedron $\mathcal{CO}$. The twelve 1-symmetric edges must correspond to diagonals of squares of the cuboctahedron. The twenty-four 2-symmetric edges either coincide with the edges of the cuboctahedron, or they connect each vertex to the antipodes of its neighbors in the cuboctahedron. Thus we obtain two possible 1-skeleta, and the operation ζ_2 carries one to the other. As graphs, these are isomorphic, (verified in Sage [27]), although the isomorphism is not induced by ζ_2 and we find it easier to relate the polyhedra with each skeleton via ζ_2 rather than the abstract graph isomorphism. For convenience, we shall refer the edges in the orbit of size 24 as 'the edges of $\mathcal{CO}$', but bear in mind that they may or may not be the orbit of edges of the convex hull of the vertex set. The edges of $\mathcal{CO}$ are solid, while those in the orbit with 12 elements are dashed in Figure 22. We will denote the 1-skeleton containing the edges of the convex cuboctahedron as CO.

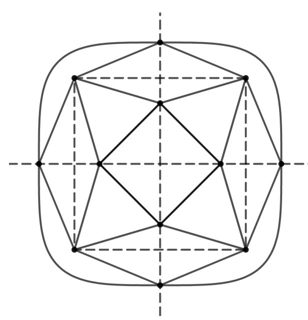


Figure 22: The graph of CO. The solid edges are one edge orbit (also referred as 'edges of $\mathcal{CO}$', and the dashed edges are another edge orbit. The top dashed edge is the same as the bottom one, and the left and right dashed edges are the same.

To describe the 3-symmetric faces in both possibilities of 1-skeleton, we will use the graph in Figure 22. Since ζ_2 fixes the size and type of 3-symmetric face, this presents no difficulty when classifying the polyhedra with 1-skeleton CO^{ζ_2}. In order to describe the 1-symmetric and 2-symmetric faces, we shall illustrate their images in the projective plane. With this understanding, the way the 2-symmetric edges lift depends completely on whether the 1-skeleton includes the edges of the cuboctahedron or not. The two possibilities for the lifts of the 1-symmetric edges must be considered; frequently one of them yields polygons whereas the other does not.

First we consider the face-transitive case (class 3^1).

Each face must use two solid edges in a row, followed by a dashed edge, two solid edges, etc. To specify the type of faces, it is enough to specify how many edges are skipped after traversing each edge on a given face F. Whenever two consecutive solid edges of F are two edges of the same square of $\mathcal{CO}$, then we get disconnected vertex-figures (pairs of triangles whether a consecutive pair of solid and dashed edge of F are consecutive around a common vertex or not). Any other configuration of solid-solid and solid-dashed in F yields connected vertex-figures, producing four possible types of faces: See Figure 23. These four cases can also be derived by thinking $\mathcal{CO}$ as a convex cuboctahedron, noting that each choice (out of 2 possibilities) of consecutive solid edges together with each choice of dashed edges (also out of 2 possibilities) completely determines the non-trivial symmetry of F that fixes the vertex between two given consecutive dashed edges, as well as the non-trivial symmetry fixing a given dashed edge.

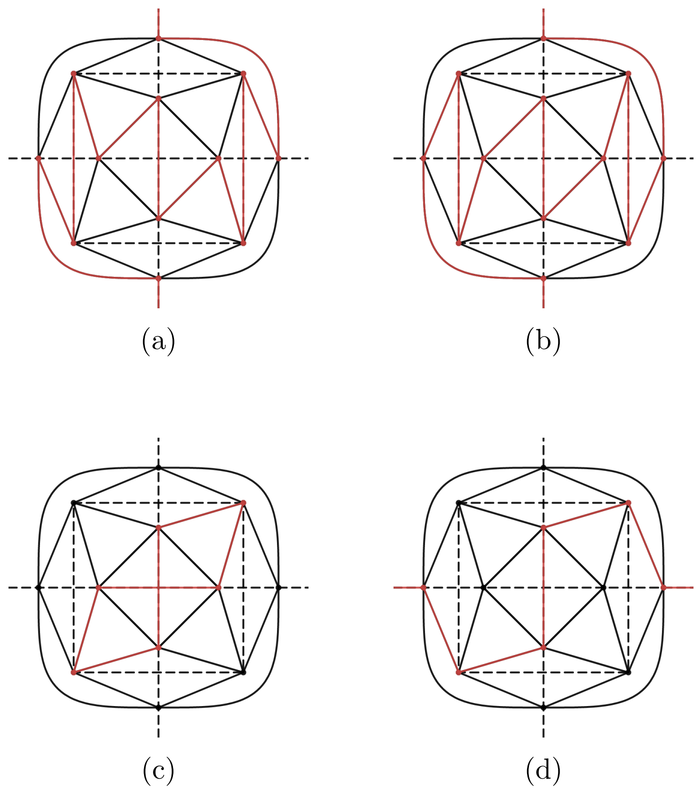


Figure 23: The four possible face-types for vertex-transitive, face-transitive 3-orbit polyhedra with 12 vertices and symmetry group $[3, 4]$, visualized on the sphere.

These polyhedra are all distinct as abstract polyhedra. In the first but not the second, every pair of adjacent edges of a face can be completed to a triangle in the graph. The third has a triangle of red vertices whereas the fourth does not. We note that though the last two polyhedra are equivelar of type $\{6, 6\}$, there is no abstract regular polyhedron of type $\{6, 6\}$ with 36 edges (see for example [14]). The first two are also not regular; the two faces meeting at a dashed edge share every third edge, whereas those meeting at a solid edge do not.

Now let us consider the face-intransitive polyhedra (in class $3^{1,2}$).

The argument used to discard two consecutive solid edges of a square of $\mathcal{CO}$ in a face works here as well. In the remaining choices the vertex-figures are connected. One type of face uses only solid edges. If the 1-skeleton is CO, when we skip no edges we get triangles, and when we skip two edges, we get hexagons, as shown in Figure 24(a) and (b). The other type of face alternates edge types; a careful analysis gives us only 3 possible faces (see Figure 24(c), (d), and (e)). For clarity, Figure 25 also shows the face in Figure 24(e) but visualized on the sphere. Each combination of 1-symmetric and 2-symmetric face yields a polyhedron.

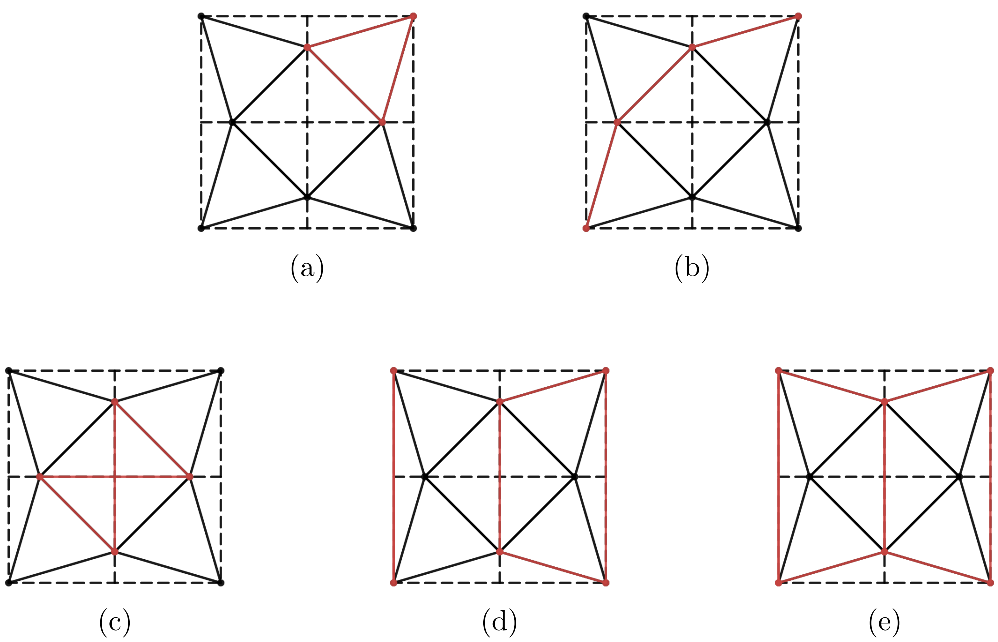


Figure 24: The possible faces for a 3-orbit polyhedron in class $3^{1,2}$ with 12 vertices and symmetry group $[3, 4]$, visualized in the projective plane. Above: The possible 1-symmetric faces. Below: The possible 2-symmetric faces.

Two polyhedra with the same kind of vertex-figure, but distinct kind of 4-gonal faces are non-isomorphic, since in the second kind of square there is a common neighbor of all vertices, but this does not occur in the first kind of square.

The details about the polyhedra with 1-skeleton CO^{ζ_2} follow similarly to those where the 1-skeleton is CO.

We summarize the discussion above in the following result.

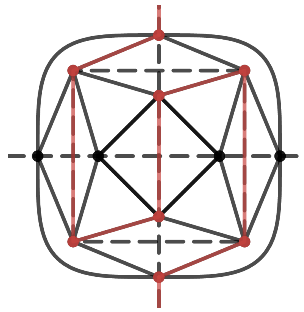


Figure 25: One possible 2-symmetric face for a 3-orbit polyhedron in class $3^{1,2}$ with 12 vertices and symmetry group $[3,4]$, visualized on a sphere.

Proposition 7.6. *Up to similarity there are* 20 *vertex-transitive* 6*-valent* 3*-orbit polyhedra in* $\mathbb{E}^3$ *with symmetry group* $[3,4]$*. The* 1*-skeleton* CO *of* 10 *of them is that of the cuboctahedron together with the two diagonals of every square; the* 1*-skeleton of the remaining* 10 *is the image of* CO *under* ζ_2 *(the* 2*-symmetric edges are those of the cuboctahedron). Four polyhedra with each* 1*-skeleton are in class* 3^1*; a sample face of each with* CO *as* 1*-skeleton is shown in Figure 23. The remaining* 12 *are in class* $3^{1,2}$*; the* 1*-symmetric and* 2*-symmetric faces of those with* CO *as* 1*-skeleton are shown in Figure 24.*

Propositions 7.5 and 7.6 are summarized in the following result.

Theorem 7.7. *Up to similarity and vt-equivalence, there are 40 vertex-transitive* 3*-orbit polyhedra with symmetry group* $[3,4]$*, summarized in Table 6.*

8 Rotational icosahedral group

Information about $[3,5]^+$	
Description	Orientation-preserving symmetry group of a dodecahedron $\mathcal{D}$
Order	60
Admissible vertex orbits	30
Involutions	15 half-turns with mirrors that join midpoints of opposite edges of $\mathcal{D}$

By Theorem 4.2, $\mathcal{P}$ cannot be in class $3^{0,1}$, and so we will assume that $\mathcal{P}$ is vertex-transitive.

Vertex-transitive polyhedra with group $[3,5]^+$	
1-symmetric edges	15
2-symmetric edges	30
Vertices	thirty 3-valent

As in the situation when the symmetry group was $[3,4]^+$, the 1-symmetric edges must join a vertex with its antipode, since these are the only line segments between vertices of $\mathcal{P}$ that are fixed pointwise by an involution in $[3,5]^+$.

The 2-symmetric edges of $\mathcal{P}$ must be stabilized by involutions that swap their endpoints. Such involutions must be half-turns. Given a vertex v_0 of $\mathcal{P}$ there are two half-turns in $[3,5]^+$ that map it to its antipode and one that fixes it. The remaining 12 half-turns are divided into three conjugacy classes under the stabilizer of v_0 in $[3,5]$. When constructing an edge with endpoints in v_0 and its image under a half-turn, two distinct choices of half-turns in the same class yield isometric collections of orbits of line segments under $[3,5]^+$, in a left- and right-handed version. Therefore there are three essentially distinct possibilities for the 2-symmetric edges of $\mathcal{P}$. In Figure 26 (a) the square represents v_0, while the black dots are midpoints of edges in the intersections of the axes of representatives of the three classes of half-turns with $\mathcal{D}$. The corresponding 2-symmetric edges are illustrated in Figure 26 (b) in solid lines, whereas the dashed line indicates the 1-symmetric edge at v_0.

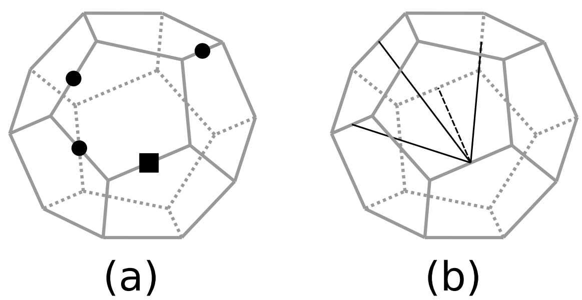


Figure 26: Kinds of edges of vertex-transitive 3-orbit polyhedra with symmetry group isomorphic to $[3,5]^+$

Since the vertices of $\mathcal{P}$ are trivalent, the 2-symmetric edges of $\mathcal{P}$ form cycles. For one of the choices of half-turns stabilizing a 2-symmetric edge these cycles are triangles; one such triangle is shown in Figure 27 (a). For the remaining two choices of half-turns the cycles are pentagons; their vertices are the midpoints of every other edge in the same Petrie path of $\mathcal{D}$. One such pentagon is convex (see Figure 27 (b)) whereas the other one is star-shaped (see Figure 27 (c)). Note that the centers of all these polygons coincide with the center of $\mathcal{D}$.

It is possible to verify directly that if the 2-symmetric edges of $\mathcal{P}$ induce triangles then the 1-skeleton of $\mathcal{P}$ is isomorphic to the truncated hemi-dodecahedron, whereas if these edges induce pentagons then the 1-skeleton of $\mathcal{P}$ is isomorphic to the truncated hemi-icosahedron. These two graphs can be obtained by identifying antipodes of the 1-skeletons of the truncated dodecahedron and the truncated icosahedron, respectively. These two graphs are shown in Figure 28 embedded in the projective plane (antipodes of the dotted circle are identified). The thick black lines represent the 1-symmetric edges, whereas the 2-symmetric ones are shown as thin gray segments.

Alternatively, the isomorphism of the 1-skeleton of $\mathcal{P}$ with the 1-skeleton of the truncated

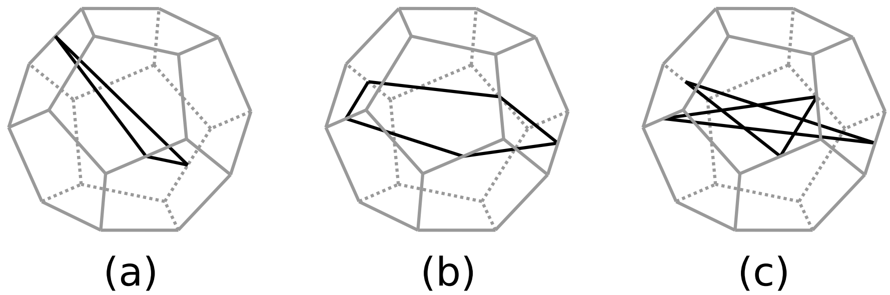


Figure 27: Polygons of 2-symmetric edges of vertex-transitive 3-orbit polyhedra with symmetry group isomorphic to $[3,5]^+$

hemi-dodecahedron or truncated hemi-icosahedron can be verified by noting that the girth of the 1-skeleton must be 3 if the 2-symmetric edges form triangles, and 5 if these edges form pentagons (note that the two antipodes of the endpoints of an edge cannot be joined by an edge since $[3,5]^+$ does not contain the central inversion). According to [22], there is only one connected vertex-transitive cubic graph with 30 vertices and girth k for each $k \in \{3,5\}$; these two graphs cannot be other than the 1-skeleton of the truncated hemi-dodecahedron and that of the truncated hemi-icosahedron.

Note the similarity to what happened when the symmetry group was $[3,4]^+$; it was also the case there that the only possible 1-skeleton was isomorphic to the 1-skeleton of the truncated hemi-cube. It is not clear to us if there is a deeper reason for this coincidence. In particular, we cannot find a suitable geometric reason why this must be the case.

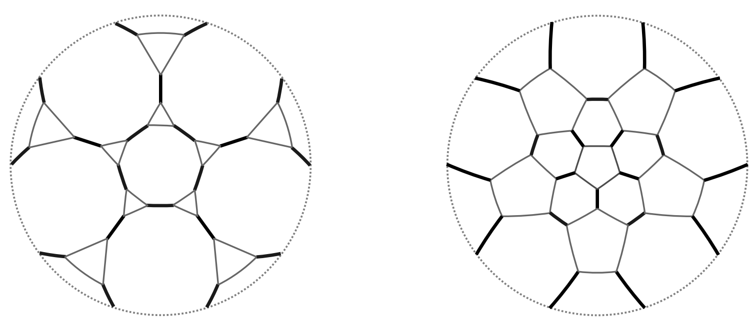


Figure 28: The truncated hemi-dodecahedron and the truncated hemi-icosahedron

The 1-skeleton of the truncated hemi-dodecahedron has 120 automorphisms. they are the 60 automorphisms of the truncated hemi-dodecahedron as a map on the projective plane, together with other 60 automorphisms that transform the contractible 10-gons into essential 10-gons. The two kind of 10-gons correspond to the pentagons and Petrie paths of the hemi-dodecahedron. On the other hand, the 1-skeleton of the truncated hemi-icosahedron has 60 automorphisms; these correspond precisely to the automorphisms of the truncated hemi-icosahedron as a map on the projective plane.

Once that we know the geometry and combinatorics of the possible 1-skeleta of $\mathcal{P}$, it only remains to describe its faces. The 1-symmetric faces must be the triangles or pentagons shown in thin lines in Figure 28; the possibilities for 2-symmetric and 3-symmetric faces are described next.

The 2-symmetric polygons of the truncated hemi-dodecahedron are necessarily 10-gons. However, their boundaries (as maps on the projective plane) may be inessential or essential, like those in Figure 29 (a), (b). The 2-symmetric polygons of the truncated hemi-icosahedron may be inessential hexagons or essential decagons. Figure 30 shows the 3-symmetric faces in the 1-skeleta of the truncated hemi-dodecahedron and truncated hemi-icosahedron. In the first case they are 15-gons, and in the second they may be either 15-gons or 9-gons.

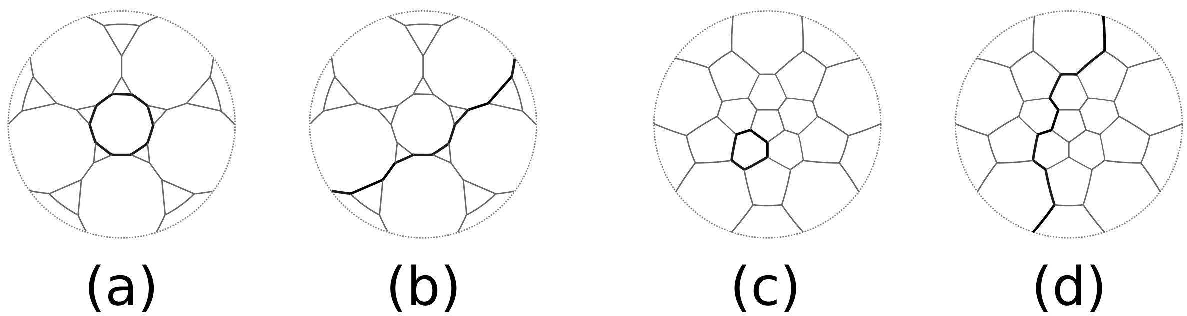


Figure 29: 2-symmetric faces

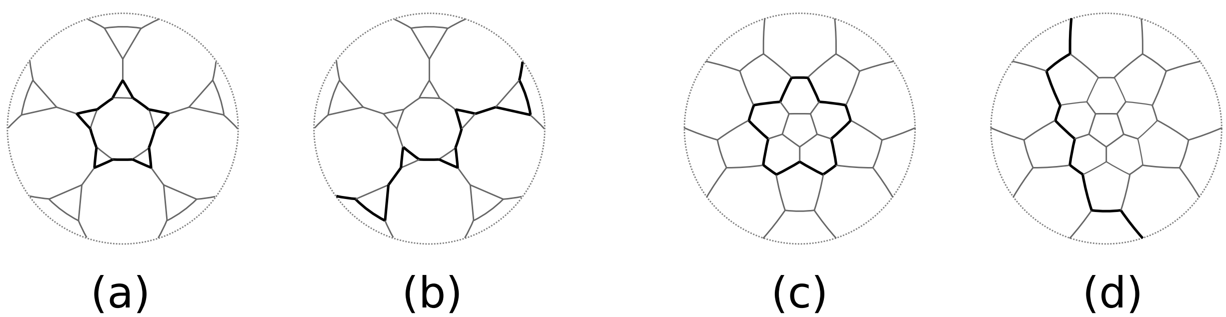


Figure 30: 3-symmetric faces

We have shown that for each symmetry type of vertex-transitive 3-orbit polyhedra there are three possible 1-skeleta, and for each choice of 1-skeleton there are two distinct choices of 2- or 3-symmetric faces. This yields 6 polyhedra in class 3^1 and 6 more in class $3^{1,2}$.

Three consecutive edges of a polygon of each of the 6 polyhedra in class 3^1 are shown in Figure 31, where thin edges are 1-symmetric and fat edges are 2-symmetric. Figures (a) to (d) represent edges of 15-gons; the remaining edges of each polygon are obtained by rotating the 3 edges around the axis between the centers joining the front and back pentagons. Figures (e) and (f) represent edges of 9-gons; the remaining edges of each polygon are obtained by rotating the 3 edges around the axis between the fat vertex and its antipode.

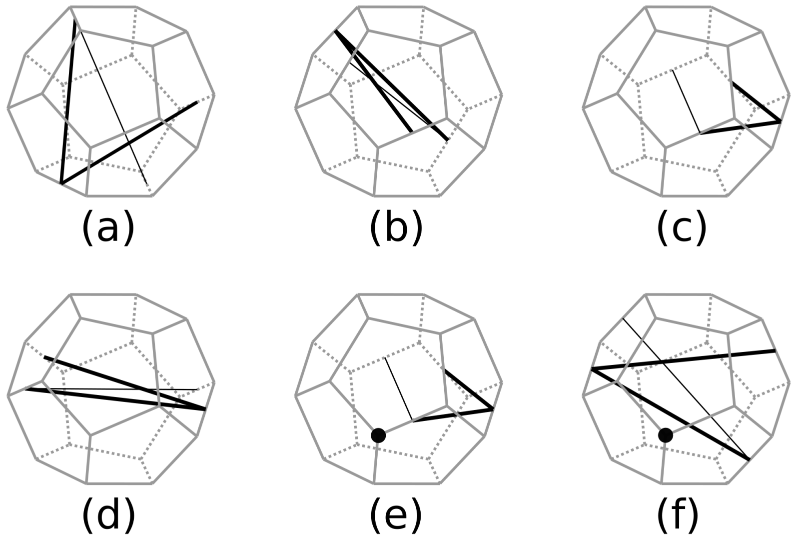


Figure 31: Three edges of four 3-symmetric 15-gons and two 3-symmetric 9-gons of polyhedra with symmetry group $[3,5]^+$

Now consider the 6 polyhedra in class $3^{1,2}$. The 2-symmetric faces of four of them are 10-gons, whereas the remaining two have hexagonal 2-symmetric faces. Two consecutive edges of three of the 10-gons are illustrated in Figure 32 (a), (b), (c); the remaining edges can be recovered through the 5-fold rotation around the axis joining the centers of the front and back pentagons. The remaining 10-gon is shown in (d). The two hexagons are those in (e) and (f), where the fat vertex indicates the axis of 3-fold rotation. In all cases, fat edges are 2-symmetric and thin edges are 1-symmetric.

Theorem 8.1. *Up to similarity there are twelve* 3*-orbit polyhedra in* $\mathbb{E}^3$ *with* $G(\mathcal{P}) = [3,5]^+$*, summarized in Table 7. Each polyhedron occurs in a left-handed and right-handed form. In all cases the convex hull of their vertex sets is an Archimedean icosidodecahedron.*

9 Full icosahedral group

Information about $[3,5]$	
Description	Symmetry group of a dodecahedron $\mathcal{D}$
Order	120
Admissible vertex orbits	12, 20, 30, 60
Involutions	Central inversion 15 plane reflections 15 half-turns with mirrors that join midpoints of opposite edges of $\mathcal{D}$

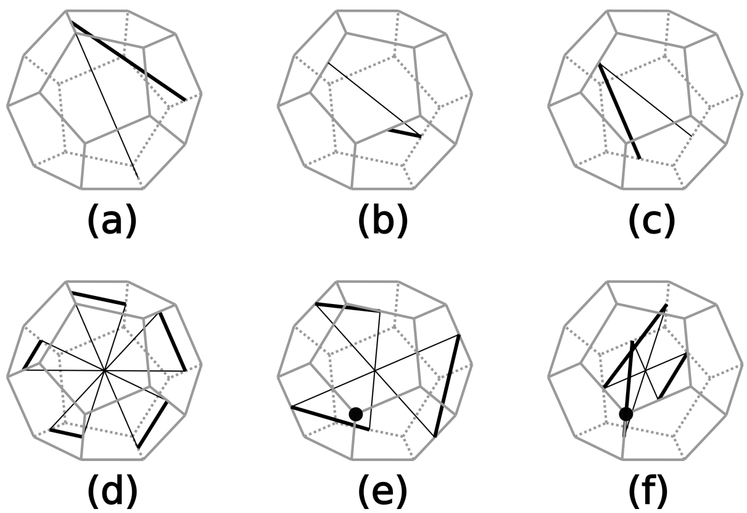


Figure 32: Two edges of three 2-symmetric 10-gons, one 10-gon and two 6-gons of polyhedra with symmetry group $[3,5]^+$

9.1 Class $3^{0,1}$

Vertex-intransitive polyhedra with group $[3,5]$	
1-symmetric edges	30
2-symmetric edges	60
1-symmetric vertices	twelve 5-valent or twenty 3-valent
2-symmetric vertices	twelve 10-valent, twenty 6-valent, or thirty 4-valent

We first consider 3-orbit polyhedra in class $3^{0,1}$. Let $\mathcal{P}$ be one such polyhedron.

The 1-symmetric vertices either lie on the 6 axes of 5-fold rotations or on the 10 axes of 3-fold rotations; in both cases, there are two vertices in each axis.

There cannot be thirty 2-symmetric vertices, lying on the axes of 2-fold rotations, since in that case only the line segments between antipodal vertices would be invariant under a subgroup of $[3,5]$ with four elements. Then there would be only one 1-symmetric edge incident to each 2-symmetric vertex, contradicting Lemma 2.1. Therefore the 2-symmetric vertices are either 10-valent (12 of them) or 6-valent (20 of them).

9.1.1 2-symmetric vertices on 5-fold axes

We first consider the case when the 2-symmetric vertices lie on the axes of 5-fold rotations of $[3,5]$. In this case we may think that they are the centers of the faces of $\mathcal{D}$. Five of the ten neighbours of each of these vertices are also 2-symmetric. These five neighbours must form an orbit under the action of the stabilizer in $[3,5]$ of the vertex. It follows that there are exactly two possible choices of 1-symmetric edges of $\mathcal{P}$, and the graph embedded in $\mathbb{E}^3$ induced by the 2-symmetric vertices is isometric to the 1-skeleton of either an icosahedron

$\{3, 5\}$, or a great icosahedron $\{3, 5/2\}$. These are isomorphic as graphs, so for the moment we will assume that the graph is embedded as the 1-skeleton of an icosahedron.

For convenience, we label the vertices of an icosahedron as in the left of Figure 33.

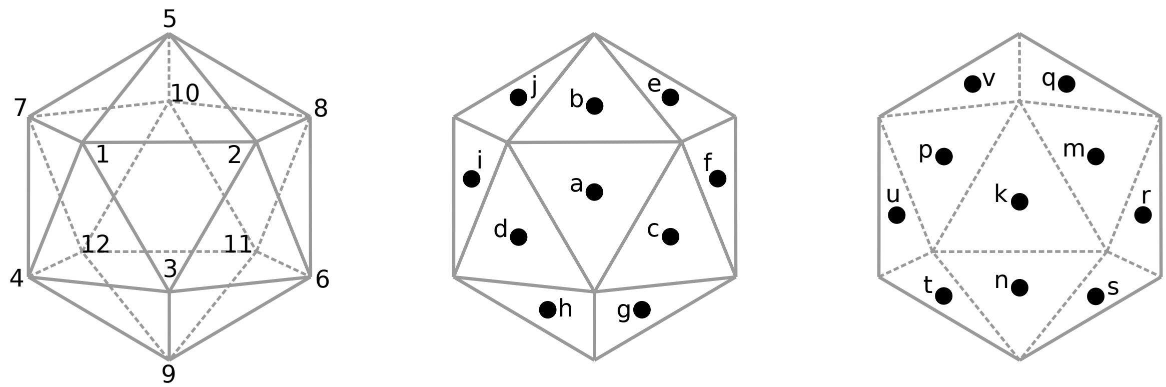


Figure 33: Labels of the vertices and faces of an icosahedron

If the 1-symmetric vertices lie on the axes of k-fold rotations of $[3, 5]$ then they are k-valent and are the vertices of a dodecahedron if $k = 3$, and of an icosahedron if $k = 5$. In either situation the neighbors of a 1-symmetric vertex v are an orbit under the stabilizer of v in $[3, 5]$. If $k = 5$ there are two such choices, but they yield vi-equivalent polyhedra. Hence, we may assume that v is adjacent to the five 2-symmetric vertices adjacent to the 2-symmetric vertex in the same ray as v from the center of $\mathcal{D}$. In that case we shall still use the numbers at the left of Figure 33 for 1-symmetric vertices, but with a prime. On the other hand, if $k = 3$ we shall locate the 1-symmetric vertices at the centers of the faces of the icosahedron induced by the 2-symmetric vertices, and use the labels in the center of Figure 33 for the centers of the front faces, and those in the right of the same figure for the centers of the faces at the back. In that situation there are four choices, yielding two non-equivalent polyhedra: one has as 1-skeleton that of the triakis icosahedron, $K_{[3,5]}$, while the other has the graph $M_{[3,5]}$ obtained from the icosahedron by adding 20 trivalent vertices at the centers of the triangles, together with the 60 edges in the orbit of $\{a, 4\}$ under $[3, 5]$. (The 'M' here stands for 'modified Kleetope'; since the construction is similar to the ordinary Kleetope construction.) A given 1-symmetric vertex is connected to the vertices of a large equilateral triangle like the one shown in Figure 34 (a). In other words, it suffices to find all polyhedra in class $3^{0,1}$ in the 1-skeleton of the triakis icosahedron $K_{[3,5]}$ and in $M_{[3,5]}$ if $k = 3$, and in the cloned 1-skeleton of the icosahedron $Cl_{[3,5]}$ if $k = 5$. Each such polyhedron will have two realizations up to vi-equivalence; one where the 1-symmetric edges are those of an icosahedron and one where they are those of a great icosahedron.

Let us remark on the effect that ζ_1 has on the 1-skeleta. For each 1-skeleton, ζ_1 changes the 1-symmetric edges from the edges of an icosahedron to those of a great icosahedron, and vice-versa. Applying ζ_1 to $Cl_{[3,5]}$ yields an isomorphic graph that can be understood as a cloned great icosahedron $Cl_{[3,5/2]}$. Applying ζ_1 to $K_{[3,5]}$ yields a 1-skeleton that is isomorphic

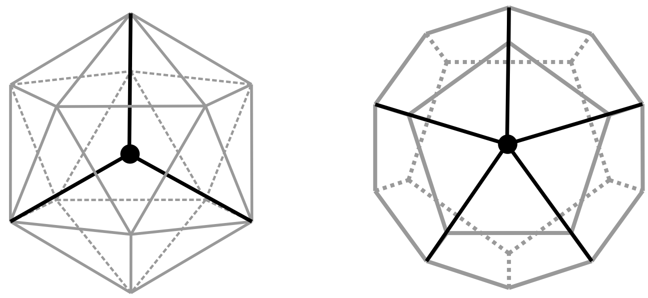


Figure 34: 1-symmetric vertex of $\mathcal{P}$

as a graph to $M_{[3,5]}$; we will call the result $M_{[3,5/2]}$ since it is obtained from the 1-skeleton of $\{3,5/2\}$ in the same way as $M_{[3,5]}$ is obtained from the 1-skeleton of $\{3,5\}$. Similarly, applying ζ_1 to $M_{[3,5]}$ yields a 1-skeleton that is isomorphic as a graph to $K_{[3,5]}$ and which we will call $K_{[3,5/2]}$. Thus we see that, for example, polyhedra with 1-skeleton $K_{[3,5]}$ also have a realization with 1-skeleton $M_{[3,5]}^{\zeta_1} = K_{[3,5/2]}$.

Recall that 3-symmetric faces have two non-equivalent automorphisms acting like reflections. Those preserving a vertex v must be plane reflections, since they must fix the center of the polyhedron, the vertex v, and the midpoint between the two neighbors v in that face. (Note that in the current circumstances those three points cannot be collinear.) On the other hand, those preserving a 1-symmetric edge while interchanging its endpoints may be plane reflections or half-turns. Let f_0 be a face of $\mathcal{P}$, let $T_1 \in G(\mathcal{P})$ fixing f_0 and a vertex v_0 of f_0, and let $T_2 \in G(\mathcal{P})$ fixing an edge e_0 while interchanging its endpoints. Assume that v_0 is adjacent to a vertex of e_0 so that T_1 and T_2 generate the stabilizer of f_0.

The automorphism T_1T_2 is Id or a rotation if T_2 is a plane reflection, and a rotatory reflection if T_2 is a half-turn.

Lemma 9.1. *The symmetry T_1T_2 is either the identity, a rotation of order 2 or a rotatory reflection of order 2.*

Proof. Assume to the contrary that T_1T_2 is none of the symmetries in the statement. The above argumentation shows that T_1T_2 is either a rotation of order k or a rotatory reflection of order $2k$, for some $k \in \{3,5\}$. In the first case the face f_0 invariant under T_1 and T_2 has k vertices that are 1-symmetric and therefore it must have $2k$ vertices that are 2-symmetric; furthermore, the 2-symmetric vertices are all in some plane perpendicular to the axis of T_1T_2. However there are no sets with 6 or 10 vertices of an icosahedron in the same plane perpendicular to a rotation axis, yielding the desired contradiction.

On the other hand, if T_1T_2 is a rotatory reflection then f_0 must have $2k$ vertices that are 1-symmetric and $4k$ vertices that are 2-symmetric. In this situation the 2-symmetric vertices must be arranged in two parallel planes, and the contradiction arises again from the fact that there should be either 6 or 10 vertices of an icosahedron in the same plane, but that does not happen.

□

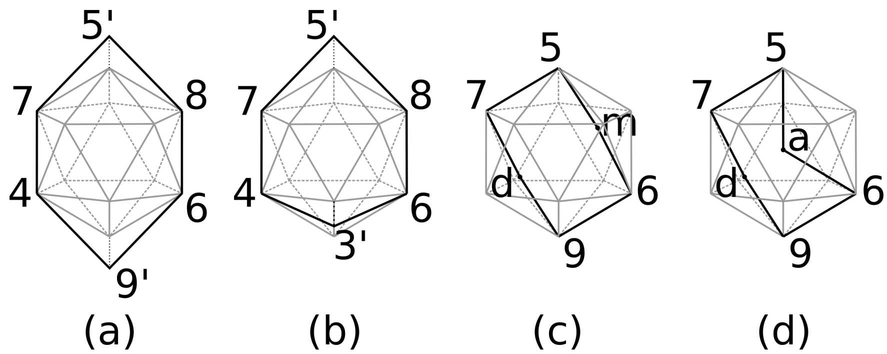


Figure 35: 3-symmetric 6-gons

If T_1T_2 is a half-turn or the central inversion then f_0 is a hexagon (of type 6_r or 6_{rh}, respectively) and its 1-symmetric edges are antipodes of each other. (For the case of the half-turn, observe that the axis ℓ is the intersection of the mirrors of T_1 and T_2, so that all 2-symmetric vertices are in a plane perpendicular to ℓ.) There is no such hexagon if the 1-skeleton of $\mathcal{P}$ is $K_{[3,5]}$, but there are two kinds if it is $Cl_{[3,5]}$ and two kinds if it is $M_{[3,5]}$. A sample hexagon of each kind with the labels of vertices given in Figure 33 is illustrated in Figure 35; (a) and (b) correspond to hexagons in $Cl_{[3,5]}$ while (c) and (d) are in $M_{[3,5]}$. (Note here that two 2-symmetric vertices of $M_{[3,5]}$ have either 2 or none common 1-symmetric neighbours.) In each case, one of these two kinds is invariant under two plane reflections and the other is invariant under one plane reflection and a half-turn. In all four cases the vertex-figures are polygons.

We claim that the four choices of $\mathcal{P}$ with hexagonal faces are mutually non-isomorphic (combinatorially). Those with $M_{[3,5]}$ as 1-skeleton have 3-valent 1-symmetric vertices while the other two have 5-valent 1-symmetric vertices. Furthermore, if the 1-skeleton of $\mathcal{P}$ is $Cl_{[3,5]}$ and the faces are invariant only under plane reflections (Figure 35 (b)) then the two 1-symmetric vertices of a face have a common neighbour, while if the faces are invariant under a plane reflection and a half-turn (Figure 35 (a)) then the 1-symmetric vertices are at distance 3 in the 1-skeleton. On the other hand, if the 1-skeleton of $\mathcal{P}$ is $M_{[3,5]}$ and f_0 is invariant under a plane reflection and a half-turn (Figure 35 (c)) then the neighbors of the 1-symmetric vertices of f_0 that are not in f_0 are antipodes in the icosahedron (and so they have no common neighbor) whereas if f_0 is invariant under two plane reflection (Figure 35 (d)) these neighbors have common neighbors among the vertices of the icosahedron.

Finally, if T_1T_2 is the identity then the faces of $\mathcal{P}$ are all the triangles consisting of two 2-symmetric edges and one 1-symmetric edge. There are no such triangles if the 1-skeleton of $\mathcal{P}$ is $M_{[3,5]}$; a triangle for each of the other candidate 1-skeleta is shown in black in Figure 36 with the notation in Figure 33. If the 1-skeleton of $\mathcal{P}$ is $K_{[3,5]}$ then $\mathcal{P}$ is combinatorially isomorphic to the triakis icosahedron. Otherwise, the 1-skeleton is $Cl_{[3,5]}$ and $\mathcal{P}$ is combinatorially a triple cover of the icosahedron, still with polygonal vertex-figures.

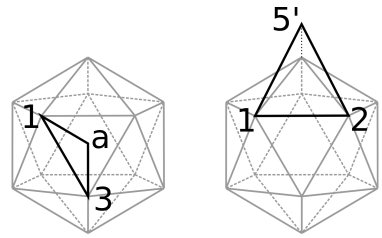


Figure 36: 3-symmetric triangles

For most of the polyhedra we obtain with a given 1-skeleton, applying ζ_1 to the polyhedron will give another polyhedron with the transformed 1-skeleton. Let us see why this does not always work. Let f_0 be a face of $\mathcal{P}$ with symmetries T_1 and T_2 as above. Consider a face f'_0 of $\mathcal{P}^{\zeta_1}$ that shares the fixed point of T_1 with f_0. Then there are symmetries T'_1 and T'_2 of f'_0 that act analogously to T_1 and T_2. In particular, T'_1 is again a plane reflection. Now, T_2 switched the endpoints of some edge $\{u, v\}$ in $\mathcal{P}$, so the analogous symmetry T'_2 must switch the endpoints of an edge $\{u, -v\}$. It follows that T'_2 is the composition of T_2 with the central inversion. Therefore, $T'_1T'_2$ is the composition of T_1T_2 with the central inversion.

Now, $T'_1T'_2$ is the identity (resp. the central inversion) if and only if T_1T_2 is the central inversion (resp. the identity). Thus ζ_1 works to interchange triangular faces with faces 6_{rh}. But if T_1T_2 is a half-turn, then $T'_1T'_2$ is a plane reflection, which is impossible. So, for example, applying ζ_1 to the two polyhedra with 1-skeleton $M_{[3,5]}$ only produces one polyhedron with 1-skeleton $K_{[3,5/2]}$. Indeed, applying ζ_1 to the polyhedron with faces as in Figure 35(d) transforms the face $(9, d, 7, 5, a, 6)$ to the walk $(9, d, 7, 9, k, 7)$ which is not a polygon.

The enumeration of the polyhedra discussed so far in this section is summarized next.

Proposition 9.2. *Up to similarity and vi-equivalence there are twelve* 3*-orbit polyhedra* $\mathcal{P}$ *in class* $3^{0,1}$ *with* $\Gamma(\mathcal{P}) = [3, 5]$ *where the* 2*-symmetric vertices are those of an icosahedron. The graph induced by the* 2*-symmetric vertices in six of them is the* 1*-skeleton of the icosahedron, and in the other six is the* 1*-skeleton of a great icosahedron.*

The 1*-skeleta of these* 12 *polyhedra are isomorphic to the triakis icosahedron, the graph* $M_{[3,5]}$ *defined above, and the cloned icosahedron. Geometrically, there is one with each of the* 1*-skeleta* $K_{[3,5]}$ *and* $K_{[3,5/2]}$*, two with each of the* 1*-skeleta* $M_{[3,5]}$ *and* $M_{[3,5/2]}$*, and three with each of the* 1*-skeleta* $Cl_{[3,5]}$ *and* $Cl_{[3,5/2]}$*.*

9.1.2 2-symmetric vertices on 3-fold axes

We move on to the case when the 2-symmetric vertices lie on the axes of 3-fold rotations of $[3, 5]$, and we may assume that they are precisely the vertices of $\mathcal{D}$. Three of the neighbors of a 2-symmetric vertex v must be 2-symmetric, and must correspond to an orbit of vertices under the stabilizer of v in $[3, 5]$. The only two possibilities are that the three 2-symmetric neighbors of v in $\mathcal{P}$ are the three neighbors of v in $\mathcal{D}$, or that they are the antipodes of the neighbors of v in $\mathcal{D}$. It follows that the subgraph of the 1-skeleton of $\mathcal{P}$ induced by the 2-symmetric vertices is the 1-skeleton of either a dodecahedron, or a great stellated dodecahedron. These two are isomorphic as graphs, and for now we will assume that the 1-skeleton is a dodecahedron. We label the vertices of the dodecahedron as in the left of Figure 37.

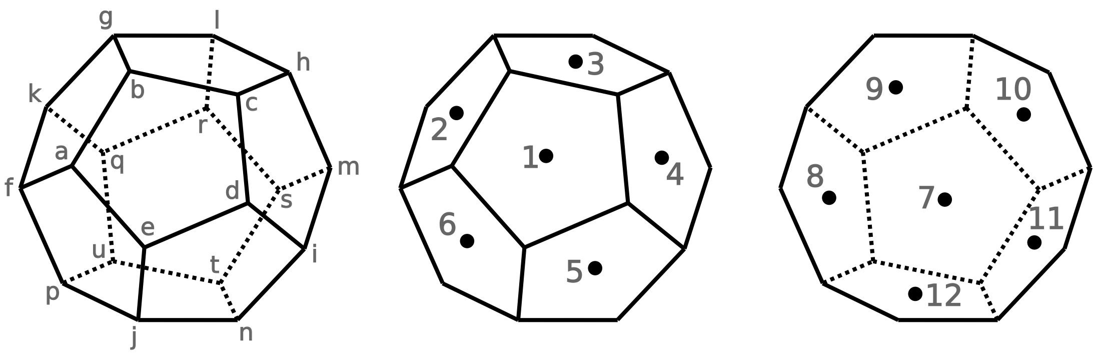


Figure 37: Labels of the vertices and faces of a dodecahedron

Once again, the 1-symmetric vertices lie on the axes of k-fold rotations of $[3, 5]$ for $k \in \{3, 5\}$. If $k = 3$ then they are trivalent and are the vertices of a dodecahedron, while if $k = 5$ they are 5-valent and are the vertices of an icosahedron. For the case where $k = 3$ we label the 1-symmetric and 2-symmetric vertices in the same ray from the center of $\mathcal{P}$ with the same letter, just distinguishing them by adding a prime to the 1-symmetric vertices. When considering the case where $k = 5$ we label the 1-symmetric vertices as in the center and right of Figure 37.

For the case $k = 3$ there are two vertex orbits of $\mathcal{D}$ of size 3 under the stabilizer of a 1-symmetric vertex, and this indicates two choices of neighbors of a 1-symmetric vertex. However, these two choices are antipodal to each other and they yield vi-equivalent polyhedra, where the 1-skeleton is the cloned dodecahedron $Cl_{[5,3]}$. On the other hand, if $k = 5$ then there are four vertex orbits of $\mathcal{D}$ of size 5 under the stabilizer in $[3, 5]$ of a 1-symmetric vertex. These give two possible 1-skeleta for $\mathcal{P}$ up to vi-equivalence: one is the 1-skeleton of the pentakis dodecahedron, denoted $K_{[5,3]}$, and the other is the 1-skeleton $M_{[5,3]}$ obtained from the 1-skeleton of the dodecahedron by adding 12 pentavalent vertices connected to the vertices of the large pentagons like the one shown in Figure 34 (b). Hence, to describe all polyhedra in class $3^{0,1}$ with full icosahedral symmetry group it suffices to find all polyhedra in class $3^{0,1}$ with 1-skeleton $Cl_{[5,3]}$, $K_{[5,3]}$, and $M_{[5,3]}$. Each such polyhedron will have two realizations up to vi-equivalence; one where the 1-symmetric edges are those of a dodecahedron and one where they are those of a great stellated dodecahedron. Indeed, the 1-skeleta where the 1-symmetric edges are those of a great stellated dodecahedron can be thought of as $Cl_{[5/2,3]}$, $K_{[5/2,3]}$, and $M_{[5/2,3]}$, which as graphs are isomorphic to $Cl_{[5,3]}, K_{[5,3]}$, and $M_{[5,3]}$, respectively. These two triples of graphs are related by the operation ζ_1 so that $Cl^{\zeta_1}_{[5,3]}, K^{\zeta_1}_{[5,3]}$, and $M^{\zeta_1}_{[5,3]}$ are isomorphic to $Cl_{[5/2,3]}$, $M_{[5/2,3]}$ and $K_{[5/2,3]}$, respectively. It follows that any polyhedron with 1-skeleton $Cl_{[5,3]}$ (resp. $K_{[5,3]}$, $M_{[5,3]}$) has another realization with 1-skeleton $Cl_{[5/2,3]}$ (resp. $M_{[5/2,3]}$, $K_{[5/2,3]}$).

As before, let T_1 and T_2 be symmetries of a face f_0 of $\mathcal{P}$ preserving a vertex and an edge, respectively. Under the current situation it is also true that T_1 is a plane reflection, whereas T_2 may be either a plane reflection or a half-turn.

The reasons explained in the proof of Lemma 9.1 still show that $\langle T_1T_2\rangle$ cannot contain a 5-fold rotation, since there is no set of 10 coplanar vertices of a dodecahedron. However, now $\langle T_1T_2\rangle$ may contain a 3-fold rotation since if $k = 3$ there are sets of 6 vertices in planes perpendicular to axes of 3-fold rotation.

If T_1T_2 is a 3-fold rotation then the three 1-symmetric edges of f_0 are coplanar. The only three such edges (up to symmetry) are shown in Figure 38 (a), where the rotation axis goes through the center of the dodecahedron and the fat vertex.

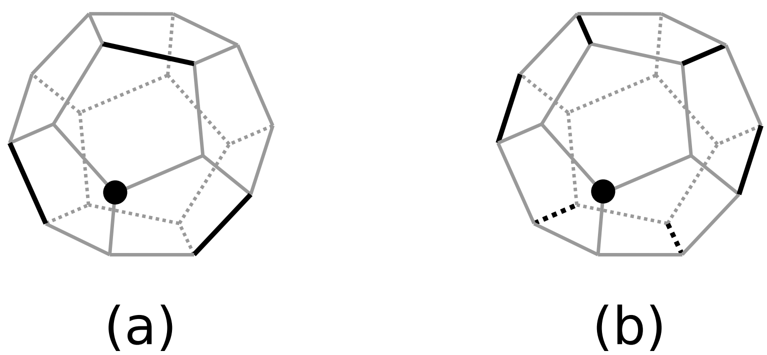


Figure 38: 1-symmetric vertex of $\mathcal{P}$

There are two ways of joining these three edges to complete a 3-symmetric 9-gon of type 9_r invariant under T_1T_2. One way is by making each of the 1-symmetric vertices adjacent to two non-adjacent consecutive vertices in the (hexagonal) convex hull of the edges, while the other is by making those vertices adjacent to opposite vertices in the convex hull of the edges. The first way can be achieved in $K_{[5,3]}$, $M_{[5,3]}$ and $Cl_{[5,3]}$, each with only one choice of 1-symmetric vertices. These are shown in that order in the first row of Figure 39. The second way can only be achieved in the graph $M_{[5,3]}$, but with two essentially distinct choices of 1-symmetric vertices. (Note that two 2-symmetric vertices of $M_{[5,3]}$ that do not belong to the same pentagon of $\mathcal{D}$ have either 2 or 0 common 1-symmetric neighbors.) These choices are illustrated in the second row of Figure 39, where in each case only two consecutive 2-symmetric edges are added, indicating the 5-neighbors of the common 1-symmetric vertex.

Here we point out that when considering the orbit of each such polygon under $[3, 5]$ the vertex-figures arising from Figure 39 (a) and (e) become disconnected (each vertex-figure at

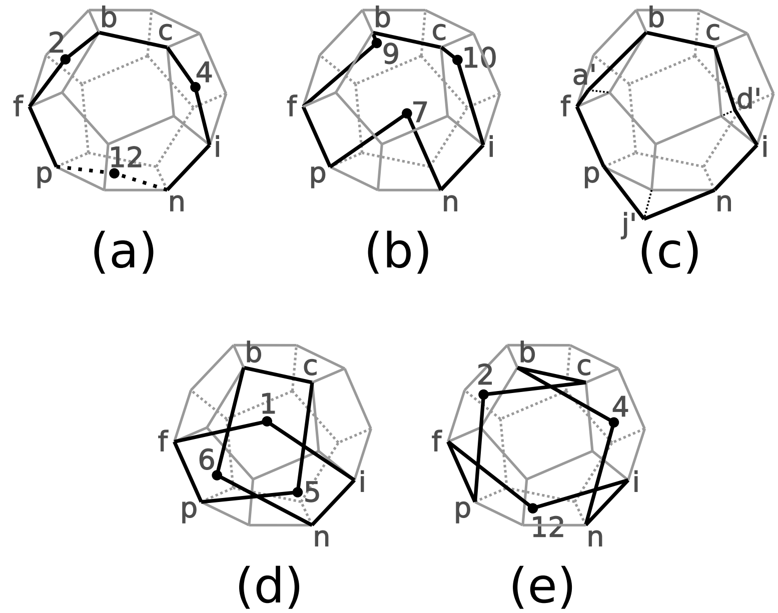


Figure 39: Possible 3-symmetric 9-gons of $\mathcal{P}$

a 2-symmetric vertex has three components, each being a line segment); in fact, Lemma 2.5 rules out the resulting structures. The vertex-figures arising from the other polygons are themselves polygons and therefore we get four 3-orbit polyhedra in class $3^{0,1}$ with 9-gonal faces and whose 2-symmetric vertices are in the axes of the 3-fold rotations.

These 3 polyhedra are mutually non-isomorphic. If the faces are (c) then the three 1-symmetric vertices of a face have a neighbor in common, and this does not happen if the faces are (b) or (d). Moreover, in the 9-gon (d) every 1-symmetric vertex has a common neighbor with each vertex of its opposite 1-symmetric edge in the 9-gon, property not satisfied by (b).

In the case that T_1T_2 is a rotatory reflection of order 6 then f_0 is an 18-gon. In this situation T_2 must be a half-turn, and the axis of T_1T_2 must belong to the mirror of T_1 and must be perpendicular to the axis of T_2. It follows that the centers of the six 1-symmetric edges must be coplanar and they must be like those in Figure 38 (b). Since the midpoints of the 1-symmetric edges lie on a plane through the center of $\mathcal{D}$, the centers of all faces coincide. Let $\mathcal{X}$ be the circle containing the centers of all edges of f_0.

In order to construct a 3-symmetric 18-gon of type 18_{rh} invariant under $\langle T_1, T_2\rangle$ and having the edges in Figure 38 (b) as its six 1-symmetric edges, any 1-symmetric vertex v_0 must be adjacent to two vertices in edges whose midpoints are consecutive in $\mathcal{X}$; furthermore, the two neighbors of a 1-symmetric vertex v_0 in f_0 must be in the same side of the plane spanned by $\mathcal{X}$. This can be done either by taking the endpoints of a 1-symmetric edge of $\mathcal{P}$ as the neighbors of v_0 in f_0, or by making v_0 adjacent to the two endpoints of a (non-trivial) diagonal of a pentagon. The first way cannot be achieved if the 1-skeleton is either $M_{[5,3]}$ or $Cl_{[5,3]}$, but it can if it is $K_{[5,3]}$ in two different manners illustrated in Figure 40. We still may

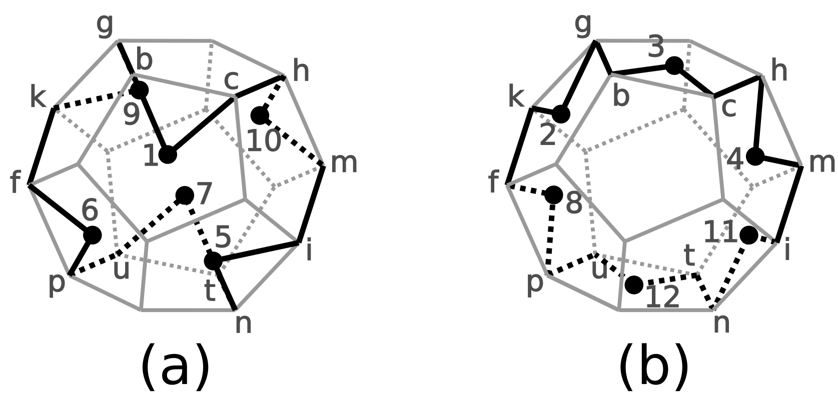


Figure 40: Two 18-gons of polyhedra in class $3^{0,1}$

discard Figure 40 (a) due to Lemma 2.5, but the vertex-figures arising from Figure 40 (b) are polygons and hence the orbit of such a face constitutes a polyhedron. The second way can be achieved in each of the three candidates of 1-skeleton, each in only one manner; these are shown in Figure 41; however, Lemma 2.5 rules out the polygons in Figure 41 (b). In the remaining two cases the corresponding vertex-figures of $\mathcal{P}$ are polygons, yielding polyhedra. Once again, the two polyhedra constructed with these 18-gons are non-isomorphic since their 1-skeleta are distinct.

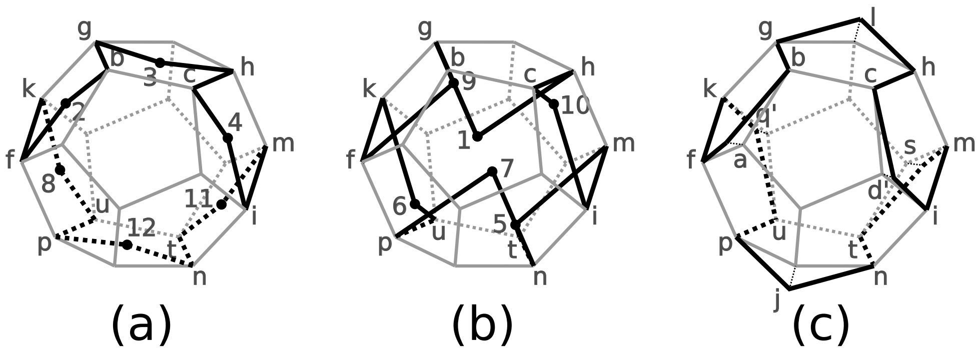


Figure 41: Three 18-gons of polyhedra in class $3^{0,1}$

We claim that the 1-symmetric edges of f_0 are antipodes if T_1T_2 is either a half-turn or the central inversion. This is clear in case T_1T_2 the central inversion. If on the other hand, it is a half-turn then T_2 is a reflection and the four 2-symmetric vertices are co-planar, which is attained in $\mathcal{D}$ if and only if the 1-symmetric edges are antipodes. Hexagons whose opposite edges are antipodes in $\mathcal{D}$ do not exist if the 1-skeleton is either $K_{[5,3]}$ or $Cl_{[5,3]}$. However, there is one orbit under $[3,5]$ of each type in the graph $M_{[5,3]}$, shown in the left and center of Figure 42. Both kinds of polygons induce connected vertex-figures.

Finally, if T_1T_2 is the identity then the f_0 is a triangle containing one 1-symmetric edge and one 1-symmetric vertex. This can only happen if the 1-skeleton of $\mathcal{P}$ is $K_{[5,3]}$, as shown

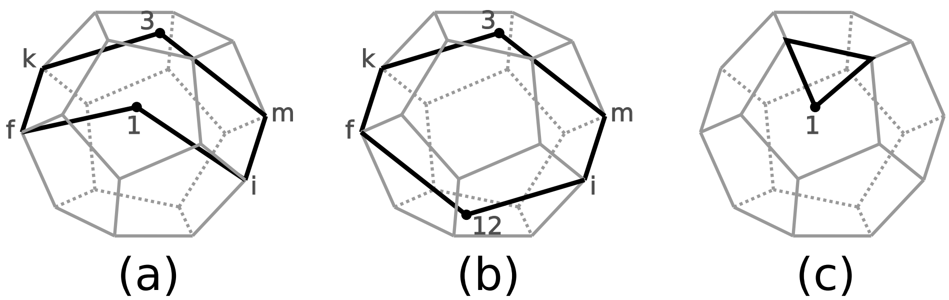


Figure 42: Two 6-gons and a triangle of polyhedra in class $3^{0,1}$

in the right of Figure 42, and the resulting polyhedron is the pentakis dodecahedron.

As in the previous subsection, applying ζ_1 to each polyhedron will give another polyhedron except in the case where T_1T_2 is a half-turn (and the faces are type 6_r). The new two possibilities for T_1T_2 (namely, a rotation of order 3 and a rotatory reflection of order 6) are interchanged by ζ_1.

Proposition 9.3. *Up to similarity and vi-equivalence there are eighteen 3-orbit polyhedra $\mathcal{P}$ in class $3^{0,1}$ with $\Gamma(\mathcal{P}) = [3,5]$ where the 2-symmetric vertices are those of a dodecahedron. The graph induced by the 2-symmetric vertices in 9 of them is the 1-skeleton of the dodecahedron, and in the other 9 is the 1-skeleton of a great stellated dodecahedron.*

The 1-skeleta of these polyhedra are isomorphic to the pentakis dodecahedron, the graph $M_{[5,3]}$ defined above, and the cloned dodecahedron. Geometrically, there are three with each of the 1-skeleta $K_{[5,3]}$ and $K_{[5/2,3]}$, two with each of the 1-skeleta $Cl_{[5,3]}$ and $Cl_{[5/2,3]}$, and four with each of the 1-skeleta $M_{[3,5]}$ and $M_{[3,5/2]}$.

We summarize our previous discussion in the following theorem.

Theorem 9.4. *Up to similarity and vi-equivalence there are thirty 3-orbit polyhedra in class $3^{0,1}$ with $\Gamma(\mathcal{P}) = [3,5]$, summarized in Table 8.*

9.2 Vertex-transitive case

Vertex-transitive polyhedra with group $[3,5]$	
1-symmetric edges	30
2-symmetric edges	60
Vertices	twenty 9-valent, thirty 6-valent, or sixty 3-valent

Now we assume that $\mathcal{P}$ is a vertex-transitive polyhedron with $G(\mathcal{P}) = [3,5]$.

We claim that $\mathcal{P}$ cannot have 30 vertices. Assume to the contrary that the vertices of $\mathcal{P}$ are in the midpoints of the edges of $\mathcal{D}$. Then $\mathcal{P}$ has thirty 6-valent vertices. Each vertex v of $\mathcal{P}$ must be incident to two 1-symmetric edges and to four 2-symmetric edges. Each

1-symmetric edge must be invariant under 4 distinct symmetries; however, the only segment between v and another midpoint of edge of $\mathcal{D}$ that has a stabilizer of order greater than 2 is the segment to its antipode. This contradicts the fact that there are two 1-symmetric edges incident to v.

Hence we only need to consider polyhedra $\mathcal{P}$ with 20 and 60 vertices.

First we consider the case where $\mathcal{P}$ has 60 vertices. In this situation the stabilizer of each vertex is generated by a plane reflection and the results of Subsection 4.3 apply. As a consequence we have the following result.

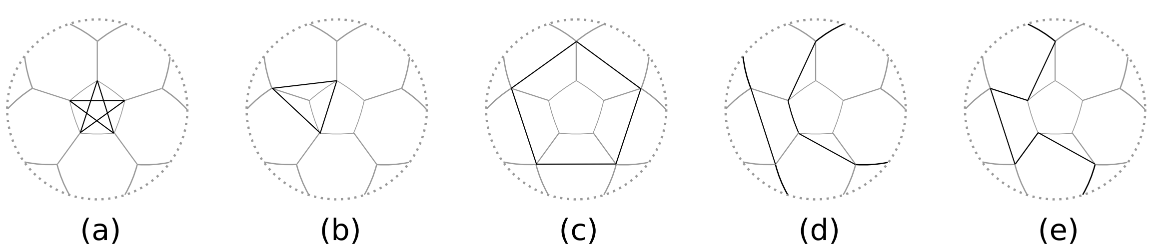
Proposition 9.5. *Up to similarity and vt-equivalence there are 48 vertex-transitive* 3*-valent* 3*-orbit polyhedra in* $\mathbb{E}^3$ *with symmetry group* $[3,5]$*:*

- *the truncations of the dodecahedron* $\{5,3\}$*, icosahedron* $\{3,5\}$*, great stellated dodecahedron* $\{5/2,3\}$*, great icosahedron* $\{3,5/2\}$*, great dodecahedron* $\{5,5/2\}$*, small stellated dodecahedron* $\{5/2,5\}$ *and the truncations of the Petrials of those* 6 *polyhedra (*12 *in total); all in class* $3^{1,2}$*;*

- *the image of the* 12 *polyhedra in the previous item under* ζ_2*, all in class* $3^{1,2}$*;*

- *the Petrials of the* 24 *polyhedra in the previous two items, all in class* 3^1*.*

We are left with the case when $\mathcal{P}$ has twenty vertices. They must be located on the axes of 3-fold rotations of $\mathcal{D}$. Then every vertex v is incident to six 2-symmetric edges and to three 1-symmetric edges. Each of these two sets of edges is an orbit under the vertex stabilizer in $[3,5]$; recall that the latter is the dihedral group D_3 with 6 elements. As mentioned in Section 4, we may visualize $\mathcal{P}$ as a graph in $\mathbb{S}^2$ with prescribed faces (cycles), and project it to $\mathbb{P}^2$, since $[3,5]$ contains the central inversion. In the hemi-dodecahedron there is precisely one orbit of vertices with 3 elements, and one of 6 elements under the stabilizer of v. The former consists of the neighbors of v and the latter of all vertices that are neither v nor adjacent to v. In particular, all edges of a 1-symmetric face project to diagonals of pentagons in the hemi-dodecahedron.



(a) (b) (c) (d) (e)

Figure 43: 1- and 2-symmetric faces of $\mathcal{P}$

An exhaustive search shows that in the graph described above all possible 1-symmetric faces are images under the symmetry group of the hemi-dodecahedron of those in black lines

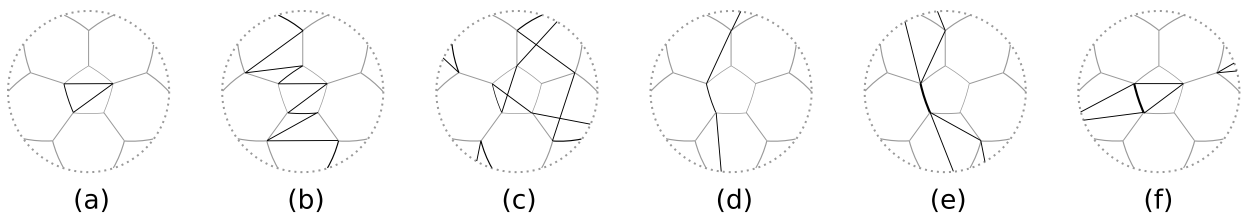


Figure 44: 3-symmetric faces of $\mathcal{P}$

in Figure 43 (a), (b) and (c); all possible 2-symmetric faces are images of those in Figure 43 (d) and (e); and all possible 3-symmetric faces are images of those in Figure 44. The polygons in Figure 44 (e), (f) are hexagons where an edge is traversed twice. When lifting them to $\mathbb{S}^2$, they become hexagons using one pair of antipodal 1-symmetric edges, and four 2-symmetric edges that are not paired by the central inversion (see Figure 48).

To convince ourselves that we obtained all possible 3-symmetric faces f_0, we divided the cases as follows. There are 3 essentially distinct ways to choose two consecutive 2-symmetric edges of f_0, illustrated in Figure 45 (a), (b) and (c). There are also 3 essentially distinct ways to choose consecutive 1-symmetric and 2-symmetric edges of f_0, shown in Figure 45 (d), (e) and (f). Finally, we have a choice on whether the stabilizer of a 1-symmetric edge contains a reflection (r) or a half-turn (h). (Recall that the stabilizer of a vertex is always a reflection.) In this way, the faces in Figure 44 (a), (b), (c), (d), (e) and (f) correspond respectively to the choices ae_r, ad_h, cd_r, cf_h, cf_r, ae_h.

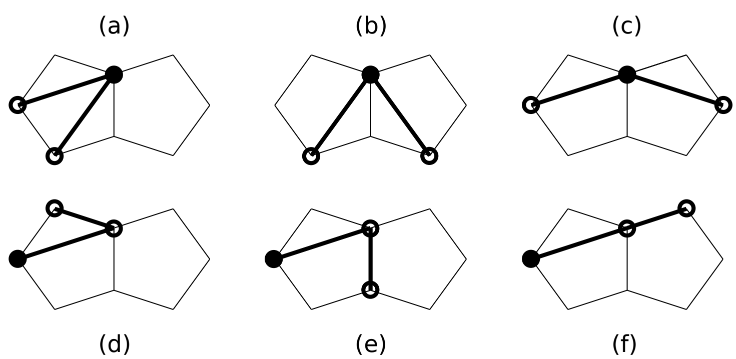


Figure 45: Cases for choices of 1- and 2-symmetric edges of a 3-symmetric face f; full and empty dots represent 1- and 2-symmetric vertices of f, respectively

Out of the 18 possible combinations in $\{a, b, c\} \times \{d, e, f\} \times \{r, h\}$ we still have to discard 12 of them. From Remark 2.9 we know that if the last entry is h then f_0 has an even number of vertices, and so it must be divisible by 6. In fact, that number must be $3k$ where k is the order of the 3-step rotation T around f_0. Since $[3, 5]$ has no isometries of order 4 we imply that T must be an involution, or a rotatory reflection of order 6. With this argument we discard bd_h, be_h, cd_h and ce_h (T would be a rotatory reflection of order 10 in all these cases).

We can further discard af_h, bf_h, ad_r, be_r and ce_r, since traversing the corresponding path in the graph on the projective plane goes more than twice through a vertex (and so it cannot be lifted as a polygon in $\mathbb{S}^2$); in ad_r all vertices belong to a pentagon appearing three times, while in the other four cases just mentioned one vertex is in the axis of the 3-step rotation T and this is a 3-fold rotation in the projective plane. The remaining three cases af_r, bd_r, bf_r induce a degenerate 15-gon in the projective plane, with the vertices of a pentagon traversed twice and all the edges of this pentagon traversed once. It is not possible to lift such a path as a polygon in $\mathbb{S}^2$ since both 2-symmetric edges at a vertex should have a vertex on some lift of the special pentagon to $\mathcal{D}$, and so two lifts to $\mathbb{S}^2$ yield degenerate 15-gons that use the vertices of a pentagon twice while the other two lift into 30-gons.

The arguments in the enumeration of the 1- and 2-symmetric faces are simpler and we omit the details.

Recall that each edge in Figures 43 and 44 can be lifted in four different ways to $\mathbb{S}^2$, by considering both antipodal pairs of vertices as candidates of endpoints. These four edges can be divided into pairs that are equivalent under $[3,5]$ and so we may think of only two possible ways of lifting the edges to $\mathbb{S}^2$.

Since 2- and 3-symmetric faces have two kinds of edges, each such face in Figures 43 and 44 can be lifted to four essentially different structures in $\mathbb{S}^2$, although they sometimes fail to be polygons. The corresponding polyhedra are related by the operations ζ, ζ_1 and ζ_2.

Figures 46, 47 and 48 illustrate respectively the 1-, 2- and 3-symmetric faces obtained by lifting those in Figures 43 and 44 in such a way that all edges are either edges of $\mathcal{D}$ or diagonals of its pentagons. We shall refer to this 1-skeleton by MD (standing for 'modified dodecahedron').

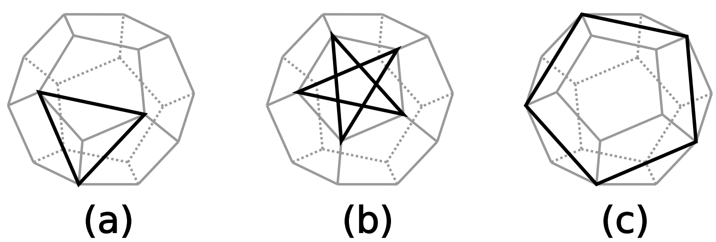


Figure 46: Candidate 1-symmetric faces of 9-valent polyhedra in class $3^{1,2}$

Moreover, there is a hexagon obtained by lifting to $\mathbb{S}^2$ the polygon in Figure 44 (f) that has no valid representative in the 1-skeleton MD. The representative in MD^{ζ_1} is shown in Figure 49.

As it can be easily seen on the triangle and pentagons in Figure 46, the operations ζ, ζ_1 and ζ_2 do not induce isomorphisms on MD, since triangles or pentagons are transformed into hexagons or decagons. In fact, MD and MD^{ζ_1} contain a clique with 5 vertices, whereas MD^{ζ} and MD^{ζ_2} do not.

According to [23] and to the software Sage [27], up to isomorphism there is only one

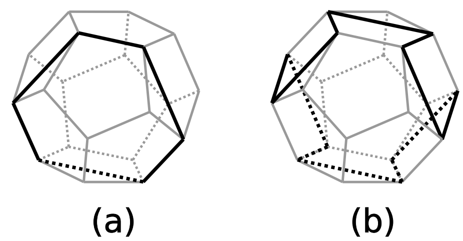


Figure 47: Candidate 2-symmetric faces of 9-valent polyhedra in class $3^{1,2}$

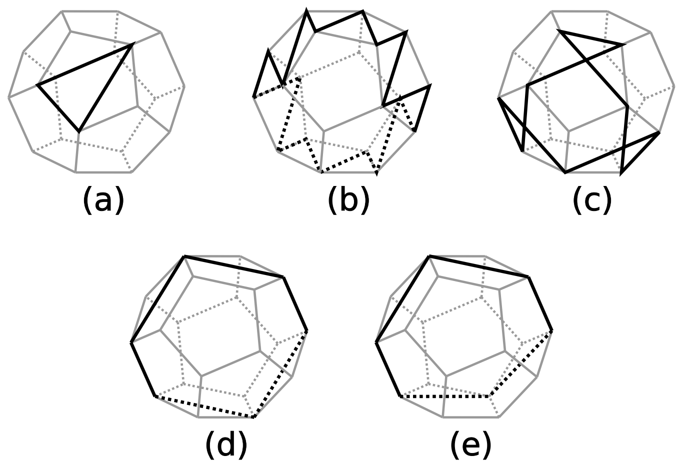


Figure 48: Candidate 3-symmetric faces of 9-valent polyhedra in class 3^1

9-valent vertex-transtive graph on 20 vertices with girth 3, diameter 3, clique number 5 and containing 140 triangles. Both MD and MD^{ζ_1} satisfy those parameters, and hence they are isomorphic. Furthermore, the 2-symmetric edges can be distinguished as those that belong to only 4 triangles while the 1-symmetric edges belong to 6 of them. It follows that every polyhedron with a realization on the 1-skeleton MD also has a realization with 1-skeleton MD^{ζ_1} and vice-versa. Since $MD^{\zeta} = (MD^{\zeta_1})^{\zeta_2}$, we can conclude that MD^{ζ} and MD^{ζ_2} are also isomorphic and serve as 1-skeleton to the same abstract polyhedra. The automorhpisms of graphs between MD and MD^{ζ_1} and between MD^{ζ_2} and MD^{ζ} are not induced by any of the operations ζ, ζ_1 or ζ_2; part of the consequence is that the polygons projecting to the projective plane to those in Figure 44 (e) and (f) are related by the isomorphism between MD and MD^{ζ_1}, but not by ζ_1 itself. Here we shall use Figures 43, 44, 46, 47, 48 and 49 as references for the enumeration of the polyhedra on each 1-skeleton.

The previous discussion and a careful examination of the sizes of the preimages of the polygons shows the following.

- The hexagon in Figure 43 (d) has four distinct polygonal preimages: the planar hexagon in Figure 47 (a) in MD, a skew hexagon in MD^{ζ}, and two skew 12-gons in MD^{ζ_1} and MD^{ζ_2}.

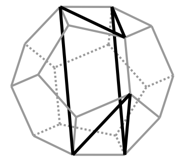


Figure 49: One additional candidate 3-symmetric face of 9-valent polyhedra in class 3^1

- The 6-gon in Figure 43 (e) has four distinct polygonal preimages: the skew 12-gon in Figure 47 (b) in MD, a planar non-convex hexagon in MD^{ζ_1}, a skew hexagon in MD^{ζ_2} and another skew 12-gon in MD^{ζ}.
- The triangle in Figure 44 (a) has four distinct polygonal preimages: the triangle in Figure 48 (a) in MD, another triangle in MD^{ζ_2}, and two skew hexagons of type 6_{rh} in MD^{ζ_1} and MD^{ζ}.
- The 9-gon in Figure 44 (b) has four distinct polygonal preimages: the 18-gon in Figure 48 (b) in MD, another 18-gon in MD^{ζ_2} (both of type 18_{rh}), and two 9-gon of type 9_r in MD^{ζ_1} and MD^{ζ}.
- The 9-gon in Figure 44 (c) has four distinct polygonal preimages: the 9-gon in Figure 48 (c) in MD, another 9-gon in MD^{ζ_2} (both of type 9_r), and two 18-gons of type 18_{rh} in MD^{ζ_1} and MD^{ζ}.
- The triangle in Figure 44 (d) has four distinct polygonal preimages: the hexagon in Figure 48 (d) in MD, another hexagon in MD^{ζ_2} (both of type 6_{rh}), and two triangles in MD^{ζ_1} and MD^{ζ}.
- The degenerate hexagon in Figure 44 (e) has only two new preimages: the hexagon in Figure 48 (e) in MD and another hexagon in MD^{ζ_2} (both of type 6_r). The remaining two preimages degenerate to unions of triangles like those in the previous item.
- The degenerate hexagon in Figure 44 (f) has only two new preimages: the hexagon in Figure 49 in MD^{ζ_1} and another hexagon in MD^{ζ} (both of type 6_r). The remaining two preimages degenerate to unions of triangles like those in the first item.

There are 20 polyhedra $\mathcal{P}$ in class 3^1 with $G(\mathcal{P}) = [3, 5]$ since in all cases the vertex-figures are polygons. They are described in Proposition 9.6. None of them are combinatorially regular; there are no regular polyhedra of type $\{3, 9\}$, $\{6, 9\}$, $\{9, 9\}$, or $\{18, 9\}$ with 360 flags (see [14]).

The 1-symmetric faces of polyhedra in class $3^{1,2}$ only have 2-symmetric edges, implying that each such face has only two non-congruent preimages in $\mathbb{S}^2$. By choosing the preimage of a 2-symmetric face we automatically obtain the preimage of all 2-symmetric edges and hence also the preimage of the 1-symmetric faces.

The three choices of 1-symmetric faces together with the two choices of 2-symmetric faces give six possible structures. However, only four of them satisfy the conditions to be a polyhedron. This is not the case when the 1-symmetric faces are like that in Figure 43 (a) and the 2-symmetric faces like that in Figure 43 (d), or if the 1-symmetric faces are like that in Figure 43 (c) and the 2-symmetric faces like that in Figure 43 (e); in those situations the vertex-figures are disconnected (union of 3 triangles). The remaining four cases indeed yield polyhedra.

Among the polyhedra in class $3^{1,2}$ only two are equivelar; in both cases the 1-symmetric faces are the orbit of the image under ζ of that in Figure 46 (a). In one polyhedron the 2-symmetric faces are the orbit of the image under ζ of that in Figure 47 (a), and in the other they are the orbit of the image under ζ_2 of that in Figure 47 (b). Those polyhedron have type $\{6, 9\}$, but they are not regular since there are no regular polyhedra with that type and 360 flags (see [14]).

Proposition 9.6. *Up to similarity there are twenty* 3*-orbit polyhedra in class* 3^1 *with* $G(\mathcal{P}) = [3, 5]$ *and* 20 *vertices. The* 1*-skeleta are* MD*,* MD^{ζ_2}*,* MD^{ζ_1} *and* MD^{ζ}*, with* 5 *polyhedra in each* 1*-skeleton.*

Up to similarity there are sixteen 3*-orbit polyhedra in class* $3^{1,2}$ *with* $G(\mathcal{P}) = [3, 5]$ *and* 20 *vertices. Their* 1*-skeleta are* MD*,* MD^{ζ_1}*,* MD^{ζ_2} *and* MD^{ζ}*, with four polyhedra in each* 1*-skeleton.*

Theorem 9.7. *Up to similarity and vt-equivalence, there are 84 vertex-transitive* 3*-orbit polyhedra with symmetry group* $[3, 5]$*, summarized in Tables 9, 10 and 11.*

10 Concluding remarks

Having determined all of the finite 3-orbit polyhedra in $\mathbb{E}^3$ with irreducible symmetry group, there are still some questions to consider, such as which polyhedra have a canonical geometric dual, and how the polyhedra are related by the Petrial operation. To help answer those questions and organize the information, we have compiled tables in Section 14. Let us describe the format of those tables.

Each table consists of polyhedra with a fixed symmetry group. The 53 polyhedra with symmetry group $[3, 4]$ are split into two tables; one for the vertex-transitive polyhedra and one for the vertex-intransitive polyhedra. The 114 polyhedra with symmetry group $[3, 5]$ are split into four tables, with the first one containing the vertex-intransitive polyhedra.

Now we explain the columns:

(a) 1**-skeleton:** The polyhedra in each table are grouped together by their 1-skeleton. The names of these 1-skeleta are introduced in the appropriate section, and they are also summarized in Table 2.

(b) **#:** We number the polyhedra with a given 1-skeleton so that other entries can cross-reference them.

(c) **Class:** The class of 3-orbit polyhedra: $3^{0,1}$, 3^1, or $3^{1,2}$.

(d) **Schläfli symbol:** For each class of polyhedra, we describe the types of faces and vertices with an extended Schläfli symbol as described in Section 2.

(e) **Petrial:** The Petrial of a given polyhedron must have the same 1-skeleton, so if the Petrial is also a polyhedron, then we indicate which by $\#N$ which means the Nth polyhedron with this same 1-skeleton. Otherwise we write N/A.

(f) **Dual:** If the polyhedron has a canonical dual (see Section 2.5), then it is recorded here. We refer to the dual by listing the table it appears in, the name of its 1-skeleton, and its number. If it has no canonical dual, but its combinatorial dual is realizable, then we denote that in the same way but starting with "(NC)" meaning "non-canonical". If the combinatorial dual is not realizable as a 3-orbit polyhedron, then we denote that with N/A.

(g) **Figure:** If there is a figure that shows the types of faces, then we refer to it here.

(h) **Type:** If there is an easy way to refer to the combinatorial type, we include it here. There are many pairs of polyhedra that are combinatorially equivalent, and if we have no other standard way to refer to the type, then we simply note here the combinatorially equivalent polyhedron by specifying the 1-skeleton and number. (For example, see the bottom two groups of Table 5.) If a polyhedron is combinatorially regular, we include that information in this column. A blank in this column indicates that the combinatorial type does not have a common name, and that no other polyhedron has the same combinatorial type.

Note that by [20, Thm. 5.1], the combinatorial type of every trivalent 3-orbit polyhedron in class $3^{1,2}$ is the truncation of a regular map. Using the GAP package RAMP ([7]), we were often able to find a single possible candidate based on the Schläfli symbol, size, and Petrial. These regular maps are denoted in one of several possible ways:

   (a) The symbol $\{p, q \mid h\}_r$ denotes the universal regular map of type $\{p, q\}$ with Petrie paths of length r and 2-holes of length h. (A 2-*hole* of a polyhedron is a walk in the 1-skeleton that, upon entering a vertex, leaves using the second exit on the left.) We may omit either r or h when the other parameters are already sufficient to distinguish the polyhedron, giving us symbols like $\{p, q\}_r$ and $\{p, q \mid h\}$.

   (b) A symbol like $\{3, 10\} * 120b$ denotes a regular polyhedron in [14].

   (c) A symbol like $N16.5$ denotes a nonorientable regular map at `https://www.math.auckland.ac.nz/~conder/RegularNonorientableMaps602.txt` while a symbol like $R3.8$ denotes an orientable regular map in `https://www.math.auckland.ac.nz/~conder/RegularOrientableMaps101.txt` (see [3]).

For the reader who wants to double-check the information in the tables, here are some helpful hints:

(a) Since the Petrial of $\mathcal{P}$ has the same 1-skeleton as $\mathcal{P}$, there are usually only a few polyhedra to check. In many cases, a possible candidate for the Petrial of $\mathcal{P}$ has faces that share many vertices in a row in $\mathcal{P}$, and so it cannot be the Petrial.

(b) Many 3-orbit polyhedra have a pair of faces that share multiple edges, which means that a geometric dual cannot exist. In particular, Propositon 2.11 rules out several possibilities. In other cases, it is enough to consider the possible Schläfli symbol of the dual and see that it does not occur among polyhedra with the appropriate symmetry group and class.

(c) Note that ζ_2 never changes the size of a 3-symmetric face or indeed any 3-symmetric walk in a 1-skeleton. So if $\mathcal{P}$ is in class $3^{1,2}$, then the Petrie polygons of $\mathcal{P}^{\zeta_2}$ are the same size as those of $\mathcal{P}$. Thus it often turns out that $(\mathcal{P}^{\zeta_2})^{\pi} = (\mathcal{P}^{\pi})^{\zeta_2}$.

(d) Perhaps the most time-consuming thing to verify is the finer structure of the faces; i.e., distinguishing n_p from n_s and distinguishing among n_r, n_h, and n_{rh}. The following observations may help:

    (a) If the symmetry group is $[3,4]^+$ or $[3,5]^+$, then the 3-symmetric faces must be type n_h.
    (b) A 3-symmetric face with n odd must be either n_r or n_h.
    (c) A 3-symmetric face n_r has its vertices contained in two parallel planes (or possibly a single plane for a certain choice of parameters in that equivalence class).
    (d) In many cases, Proposition 4.3 rules out type n_h for a 3-symmetric face.

(e) When looking for combinatorially regular polyhedra, keep in mind that if $\mathcal{P}$ is combinatorially regular, then so is $\mathcal{P}^{\pi}$. This cuts the search space down considerably, since most 3-orbit polyhedra in class 3^1 have Petrials where the two types of faces have different sizes.

We summarize some of the interesting features of our data in the final theorem.

Theorem 10.1. *There are* 188 3*-orbit polyhedra in* $\mathbb{E}^3$ *with irreducible symmetry group. There are* 5 *with symmetry group* $[3,3]$*,* 4 *with symmetry group* $[3,4]^+$*,* 53 *with symmetry group* $[3,4]$*,* 12 *with symmetry group* $[3,5]^+$*, and* 114 *with symmetry group* $[3,5]$*. Furthermore,*

(a) *Six are combinatorially regular; the rest are combinatorially* 3*-orbit.*

(b) *There are* 109 *distinct combinatorial types:* 30 *that have a single* 3*-orbit geometric realization, and* 79 *that have two* 3*-orbit geometric realizations, including all of the combinatorial types of polyhedra with symmetry group* $[3,5]$*.*

(c) *There are* 44 *polyhedra in class* $3^{0,1}$ *(vertex-intransitive),* 72 *polyhedra in class* $3^{1,2}$ *(face-intransitive), and* 72 *polyhedra in class* 3^1 *(vertex- and face-transitive).*

(d) 88 *of the polyhedra are* 3*-valent.*

(e) *There are* 36 *polyhedra (*18 *pairs) that have a canonical geometric dual. There are also two polyhedra with no canonical dual, but whose combinatorial dual is geometrically realizable as a* 3*-orbit polyhedron. None of the polyhedra are self-dual (not even combinatorially).*

(f) *There are* 130 *polyhedra (*65 *pairs) such that the Petrial is also a polyhedron.*

11 Data availability

All data generated or analysed during this study are included in this published article

12 Acknowledgments

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13 Appendix

Symmetry	Notation	Common name	Petrial	Type
$[3,3]$	$\{3,3\}$	Tetrahedron	$\{4,3\}_3$	
	$\{4,3\}_3$	Hemi-cube	$\{3,3\}$	
$[3,4]$	$\{3,4\}$	Octahedron	$\{6,4\}_3$	
	$\{4,3\}$	Cube	$\{6,3\}_4$	
	$\{6,4\}_3$	N/A	$\{3,4\}$	
	$\{6,3\}_4$	N/A	$\{4,3\}$	
$[3,5]$	$\{3,5\}$	Icosahedron	$\{10,5\}_3$	
	$\{3,\frac{5}{2}\}$	Great icosahedron	$\{\frac{10}{3},\frac{5}{2}\}_3$	Isomorphic to $\{3,5\}$
	$\{5,3\}$	Dodecahedron	$\{10,3\}_5$	
	$\{\frac{5}{2},3\}$	Great stellated dodecahedron	$\{\frac{10}{3},3\}_5$	Isomorphic to $\{5,3\}$
	$\{5,\frac{5}{2}\}$	Great dodecahedron	$\{6,\frac{5}{2}\}_5$	
	$\{\frac{5}{2},5\}$	Small stellated dodecahedron	$\{6,5\}_5$	Isomorphic to $\{5,\frac{5}{2}\}$
	$\{10,5\}_3$	N/A	$\{3,5\}$	
	$\{\frac{10}{3},\frac{5}{2}\}_3$	N/A	$\{3,\frac{5}{2}\}$	Isomorphic to $\{10,5\}_3$
	$\{10,3\}_5$	N/A	$\{5,3\}$	
	$\{\frac{10}{3},3\}_5$	N/A	$\{\frac{5}{2},3\}$	Isomorphic to $\{10,3\}_5$
	$\{6,\frac{5}{2}\}_5$	N/A	$\{5,\frac{5}{2}\}$	
	$\{6,5\}_5$	N/A	$\{\frac{5}{2},5\}$	Isomorphic to $\{6,\frac{5}{2}\}$

Table 1: The finite regular polyhedra in $\mathbb{E}^3$

1-skeleton	Description	Reference
C	Cube with subdivided edges and altitudes of each face	Section 7.1.3
$Cl_{[p,q]}$	Cloned skeleton of $\{p,q\}$	Section 4.1
CO	Skeleton of cuboctahedron with diagonals of square faces added	Section 7.2
$H_{[p,q]}$	Skeleton of the truncated hemi-$\{p,q\}$, embedded in $\mathbb{E}^3$. (Has left- and right-handed forms)	Section 6 Section 8
$K_{[p,q]}$	Skeleton of the Kleetope over $\{p,q\}$ (e.g. triakis icosahedron)	Section 2.5
$M_{[p,q]}$	Skeleton of a modified Kleetope over $\{p,q\}$	Section 9.1
MD	Skeleton of a dodecahedron with all diagonals of faces added	Section 9.2
$T_{[p,q]}$	Skeleton of the truncated $\{p,q\}$	Section 2.5

Table 2: The 1-skeleta of 3-orbit polyhedra in $\mathbb{E}^3$ with irreducible symmetry group. Whenever $\{p,q\}$ is combinatorially equivalent to $\{p',q'\}$, it is also the case that a 1-skeleton of the form $S_{[p,q]}$ is combinatorially equivalent to $S_{[p',q']}$. Additionally, $H_{[4,3]}$ is combinatorially isomorphic to $T_{[3,3]}$.

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$T_{[3,3]}$	1	$3^{1,2}$	$\{(3,6_p),3\}$	#3	$K_{[3,3]}$ #1	N/A	Tr($\{3,3\}$)
	2	$3^{1,2}$	$\{(3,8_s),3\}$	#4	N/A	N/A	Tr($\{4,3\}_3$)
	3	3^1	$\{12_{rh},3\}$	#1	N/A	N/A	Tr($\{3,3\})^\pi$
	4	3^1	$\{9_r,3\}$	#2	N/A	N/A	Tr($\{4,3\}_3)^\pi$
$K_{[3,3]}$	1	$3^{0,1}$	$\{3,(3,6)\}$	N/A	$T_{[3,3]}$ #1	N/A	Kl($\{3,3\}$)

Table 3: The five 3-orbit polyhedra with symmetry group $[3,3]$

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$H_{[4,3]}$	1	$3^{1,2}$	$\{(3,6_s),3\}$	#3	N/A	12c	Tr($\{3,3\}$)
	2	$3^{1,2}$	$\{(3,8_s),3\}$	#4	N/A	12d	Tr($\{4,3\}_3$)
	3	3^1	$\{12_h,3\}$	#1	N/A	12b	Tr($\{3,3\})^\pi$
	4	3^1	$\{9_h,3\}$	#2	N/A	12a	Tr($\{4,3\}_3)^\pi$

Table 4: The four 3-orbit polyhedra with symmetry group $[3,4]^+$. Each polyhedron comes in a left-handed and right-handed form.

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$K_{[3,4]}$	1	$3^{0,1}$	$\{3,(3,8)\}$	N/A	Table 6 $T_{[4,3]}$ #1	16a	Kl($\{3,4\}$)
	2	$3^{0,1}$	$\{6_r,(3,8)\}$	N/A	N/A	16b	
	3	$3^{0,1}$	$\{6_{rh},(3,8)\}$	N/A	N/A	16c	
$K_{[4,3]}$	1	$3^{0,1}$	$\{3,(4,6)\}$	#2	Table 6 $T_{[3,4]}$ #1	18a	Kl($\{4,3\}$)
	2	$3^{0,1}$	$\{12_{rh},(4,6)\}$	#1	N/A	18b	Kl($\{4,3\})^\pi$
$K^{\zeta_1}_{[4,3]}$	1	$3^{0,1}$	$\{6_r,(4,6)\}$	#2	N/A	18c	
	2	$3^{0,1}$	$\{6_r,(4,6)\}$	#1	N/A	18d	
	3	$3^{0,1}$	$\{6_{rh},(4,6)\}$	N/A	N/A	18e	
$Cl_{[4,3]}$	1	$3^{0,1}$	$\{6_r,(3,6)\}$	N/A	N/A	19a	
	2	$3^{0,1}$	$\{6_{rh},(3,6)\}$	N/A	N/A	19b	
C	1	$3^{0,1}$	$\{6_r,(3,4)\}$	#2	Table 6 CO #6	21a	
	2	$3^{0,1}$	$\{6_{rh},(3,4)\}$	#1	N/A	21b	
	3	$3^{0,1}$	$\{12_{rh},(3,4)\}$	N/A	N/A	21c	

Table 5: The 13 vertex-intransitive 3-orbit polyhedra with symmetry group $[3,4]$

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$T_{[3,4]}$	1	$3^{1,2}$	$\{(4_p, 6_p), 3\}$	#3	Table 5 $K_{[4,3]}$ #1	N/A	$\mathrm{Tr}(\{3,4\})$
	2	$3^{1,2}$	$\{(4_p, 12_s), 3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{6,4\}_3)$
	3	3^1	$\{12_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{3,4\})^\pi$
	4	3^1	$\{12_r, 3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{6,4\}_3)^\pi$
$T^{\zeta_2}_{[3,4]}$	1	$3^{1,2}$	$\{(4_s, 12_s), 3\}$	#3	N/A	N/A	$\mathrm{Tr}(\{6,4\}_3)$
	2	$3^{1,2}$	$\{(4_s, 6_s), 3\}$	#4	(NC) Table 5 $K_{[4,3]}$ #1	N/A	$\mathrm{Tr}(\{3,4\})$
	3	3^1	$\{12_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{6,4\}_3)^\pi$
	4	3^1	$\{12_r, 3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{3,4\})^\pi$
$T_{[4,3]}$	1	$3^{1,2}$	$\{(3, 8_p), 3\}$	#4	Table 5 $K_{[3,4]}$ #1	N/A	$\mathrm{Tr}(\{4,3\})$
	2	$3^{1,2}$	$\{(3, 12_s), 3\}$	#3	N/A	N/A	$\mathrm{Tr}(\{6,3\}_4)$
	3	3^1	$\{12_r, 3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{6,3\}_4)^\pi$
	4	3^1	$\{18_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{4,3\})^\pi$
$T^{\zeta_2}_{[4,3]}$	1	$3^{1,2}$	$\{(6_s, 8_s), 3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{4,6\}_3)$
	2	$3^{1,2}$	$\{(6_s, 6_s), 3\}$	#3	N/A	N/A	$\{6,3\}_{(2,2)}$ (reg.)
	3	3^1	$\{12_r, 3\}$	#2	N/A	N/A	$\{6,3\}^\pi_{(2,2)}$ (reg.)
	4	3^1	$\{18_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{4,6\}_3)^\pi$
$T^{\zeta}_{[4,3]}$	1	$3^{1,2}$	$\{(6_s, 12_s), 3\}$	#4	N/A	N/A	$\mathrm{Tr}(R3.8)$
	2	$3^{1,2}$	$\{(6_s, 8_s), 3\}$	#3	N/A	N/A	$\mathrm{Tr}(R3.4)$
	3	3^1	$\{9_r, 3\}$	#2	N/A	N/A	$\mathrm{Tr}(R3.4)^\pi$
	4	3^1	$\{12_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(R3.8)^\pi$
CO	1	3^1	$\{12_{rh}, 6_r\}$	N/A	N/A	23a	CO^{ζ_2} #1
	2	3^1	$\{12_h, 6_{rh}\}$	N/A	N/A	23b	CO^{ζ_2} #2
	3	3^1	$\{6_{rh}, 6_r\}$	#6	N/A	23c	CO^{ζ_2} #3
	4	3^1	$\{6_h, 6_{rh}\}$	#9	N/A	23d	CO^{ζ_2} #4
	5	$3^{1,2}$	$\{(3, 4_p), 6_r\}$	N/A	N/A	24ac	CO^{ζ_2} #8
	6	$3^{1,2}$	$\{(3, 4_p), 6_r\}$	#3	Table 5, C #1	24ad	CO^{ζ_2} #9
	7	$3^{1,2}$	$\{(3, 8_s), 6_r\}$	N/A	N/A	24ae, 25	CO^{ζ_2} #10
	8	$3^{1,2}$	$\{(6_p, 4_p), 6_{rh}\}$	N/A	N/A	24bc	CO^{ζ_2} #5
	9	$3^{1,2}$	$\{(6_p, 4_p), 6_{rh}\}$	#4	N/A	24bd	CO^{ζ_2} #6
	10	$3^{1,2}$	$\{(6_p, 8_s), 6_{rh}\}$	N/A	N/A	24be	CO^{ζ_2} #7
CO^{ζ_2}	1	3^1	$\{12_{rh}, 6_r\}$	N/A	N/A	23a	CO #1
	2	3^1	$\{12_h, 6_{rh}\}$	N/A	N/A	23b	CO #2
	3	3^1	$\{6_{rh}, 6_r\}$	#9	N/A	23c	CO #3
	4	3^1	$\{6_h, 6_{rh}\}$	#6	N/A	23d	CO #4
	5	$3^{1,2}$	$\{(6_s, 4_s), 6_r\}$	N/A	N/A	24ac	CO #8
	6	$3^{1,2}$	$\{(6_s, 4_p), 6_r\}$	#4	N/A	24ad	CO #9
	7	$3^{1,2}$	$\{(6_s, 8_s), 6_r\}$	N/A	N/A	24ae	CO #10
	8	$3^{1,2}$	$\{(3, 4_s), 6_{rh}\}$	N/A	N/A	24bc	CO #5
	9	$3^{1,2}$	$\{(3, 4_p), 6_{rh}\}$	#3	(NC) Table 5, C #1	24bd	CO #6
	10	$3^{1,2}$	$\{(3, 8_s), 6_{rh}\}$	N/A	N/A	24be	CO #7

Table 6: The 40 vertex-transitive 3-orbit polyhedra with symmetry group $[3,4]$

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$H_{[5,3]}$	1	3^1	$\{15_h, 3\}$	#4	N/A	31a	$\mathrm{Tr}(\{5,3\}_5)^\pi$
	2	3^1	$\{15_h, 3\}$	#3	N/A	31b	$\mathrm{Tr}(\{5,3\}_5)^\pi$
	3	$3^{1,2}$	$\{(3, 10_s), 3\}$	#2	N/A	27a, 32a	$\mathrm{Tr}(\{5,3\}_5)$
	4	$3^{1,2}$	$\{(3, 10_s), 3\}$	#1	N/A	27a, 32b	$\mathrm{Tr}(\{5,3\}_5)$
$H_{[3,5]}$	1	3^1	$\{15_h, 3\}$	#4	N/A	31c	$\mathrm{Tr}(\{5,5\}_3)^\pi$
	2	3^1	$\{9_h, 3\}$	#3	N/A	31e	$\mathrm{Tr}(\{3,5\}_5)^\pi$
	3	$3^{1,2}$	$\{(5, 6_s), 3\}$	#2	N/A	27b, 32e	$\mathrm{Tr}(\{3,5\}_5)$
	4	$3^{1,2}$	$\{(5, 10_s), 3\}$	#1	N/A	27b, 32d	$\mathrm{Tr}(\{5,5\}_3)$
$H_{[3,5/2]}$	1	3^1	$\{15_h, 3\}$	#4	N/A	31d	$\mathrm{Tr}(\{5,5\}_3)^\pi$
	2	3^1	$\{9_h, 3\}$	#3	N/A	31f	$\mathrm{Tr}(\{3,5\}_5)^\pi$
	3	$3^{1,2}$	$\{(5, 6_s), 3\}$	#2	N/A	27c, 32f	$\mathrm{Tr}(\{3,5\}_5)$
	4	$3^{1,2}$	$\{(5, 10_s), 3\}$	#1	N/A	27c, 32c	$\mathrm{Tr}(\{5,5\}_3)$

Table 7: The twelve 3-orbit polyhedra with symmetry group $[3,5]^+$. Each polyhedron comes in a left-handed and a right-handed form. Note that $H_{[3,5]}$ and $H_{[3,5/2]}$ are distinguished by the choice of 1-symmetric edges, see Figures 26 and 27.

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$K_{[3,5]}$	1	$3^{0,1}$	$\{3,(3,10)\}$	N/A	Table 9 $T_{[5,3]}$ #1	36a	Kl($\{3,5\}$)
$K_{[3,5/2]}$	1	$3^{0,1}$	$\{3,(3,10)\}$	N/A	Table 9 $T_{[5/2,3]}$ #1	36a	Kl($\{3,5\}$)
$M_{[3,5]}$	1	$3^{0,1}$	$\{6_{rh},(3,10)\}$	N/A	N/A	35c	$M_{[3,5/2]}$ #1
	2	$3^{0,1}$	$\{6_r,(3,10)\}$	N/A	N/A	35d	$M_{[3,5/2]}$ #2
$M_{[3,5/2]}$	1	$3^{0,1}$	$\{6_{rh},(3,10)\}$	N/A	N/A	35c	$M_{[3,5]}$ #1
	2	$3^{0,1}$	$\{6_r,(3,10)\}$	N/A	N/A	35d	$M_{[3,5]}$ #2
$Cl_{[3,5]}$	1	$3^{0,1}$	$\{3,(5,10)\}$	N/A	Table 10 $T_{[5/2,5]}$ #1	36b	Kl($\{5,5 \mid 3\}$)
	2	$3^{0,1}$	$\{6_{rh},(5,10)\}$	N/A	N/A	35a	$Cl_{[3,5/2]}$ #2
	3	$3^{0,1}$	$\{6_r,(5,10)\}$	N/A	N/A	35b	$Cl_{[3,5/2]}$ #3
$Cl_{[3,5/2]}$	1	$3^{0,1}$	$\{3,(5,10)\}$	N/A	Table 10 $T_{[5,5/2]}$ #1	36b	Kl($\{5,5 \mid 3\}$)
	2	$3^{0,1}$	$\{6_{rh},(5,10)\}$	N/A	N/A	35a	$Cl_{[3,5]}$ #2
	3	$3^{0,1}$	$\{6_r,(5,10)\}$	N/A	N/A	35b	$Cl_{[3,5]}$ #3
$K_{[5,3]}$	1	$3^{0,1}$	$\{18_{rh},(5,6)\}$	#3	N/A	40b	Kl($\{5,3\})^{\pi}$
	2	$3^{0,1}$	$\{18_{rh},(5,6)\}$	N/A	N/A	41a	$K_{[5/2,3]}$ #2
	3	$3^{0,1}$	$\{3,(5,6)\}$	#1	Table 9 $T_{[3,5]}$ #1	42c	Kl($\{5,3\}$)
$K_{[5/2,3]}$	1	$3^{0,1}$	$\{18_{rh},(5,6)\}$	#3	N/A	40b	Kl($\{5,3\})^{\pi}$
	2	$3^{0,1}$	$\{18_{rh},(5,6)\}$	N/A	N/A	41a	$K_{[5,3]}$ #2
	3	$3^{0,1}$	$\{3,(5,6)\}$	#1	Table 9 $T_{[3,5/2]}$ #1	42c	Kl($\{5,3\}$)
$Cl_{[5,3]}$	1	$3^{0,1}$	$\{9_r,(3,6)\}$	#2	Table 11 MD^{ζ_1} #3	39c	$Cl_{[5/2,3]}$ #1
	2	$3^{0,1}$	$\{18_{rh},(3,6)\}$	#1	N/A	41c	$Cl_{[5/2,3]}$ #2
$Cl_{[5/2,3]}$	1	$3^{0,1}$	$\{9_r,(3,6)\}$	#2	Table 11 MD #2	39c	$Cl_{[5,3]}$ #1
	2	$3^{0,1}$	$\{18_{rh},(3,6)\}$	#1	N/A	41c	$Cl_{[5,3]}$ #2
$M_{[5,3]}$	1	$3^{0,1}$	$\{9_r,(5,6)\}$	N/A	N/A	39b	$M_{[5/2,3]}$ #1
	2	$3^{0,1}$	$\{9_r,(5,6)\}$	#4	Table 11 MD^{ζ_1} #1	39d	$M_{[5/2,3]}$ #2
	3	$3^{0,1}$	$\{6_r,(5,6)\}$	N/A	N/A	42a	$M_{[5/2,3]}$ #3
	4	$3^{0,1}$	$\{6_{rh},(5,6)\}$	#2	N/A	42b	$M_{[5/2,3]}$ #4
$M_{[5/2,3]}$	1	$3^{0,1}$	$\{9_r,(5,6)\}$	N/A	N/A	39b	$M_{[5,3]}$ #1
	2	$3^{0,1}$	$\{9_r,(5,6)\}$	#4	Table 11 MD #4	39d	$M_{[5,3]}$ #2
	3	$3^{0,1}$	$\{6_r,(5,6)\}$	N/A	N/A	42a	$M_{[5,3]}$ #3
	4	$3^{0,1}$	$\{6_{rh},(5,6)\}$	#2	N/A	42b	$M_{[5,3]}$ #4

Table 8: The 30 vertex-intransitive 3-orbit polyhedra with symmetry group $[3,5]$

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$T_{[3,5]}$	1	$3^{1,2}$	$\{(5,6_p),3\}$	#3	Table 8 $K_{[5,3]}$ #3	N/A	$\mathrm{Tr}(\{3,5\})$
	2	$3^{1,2}$	$\{(5,20_s),3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{10,5\}_3)$
	3	3^{1}	$\{18_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{3,5\})^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{10,5\}_3)^\pi$
$T^{\zeta_2}_{[3,5]}$	1	$3^{1,2}$	$\{(10_s,12_s),3\}$	#3	N/A	N/A	$\mathrm{Tr}(N16.5)$
	2	$3^{1,2}$	$\{(10_s,10_s),3\}$	#4	N/A	N/A	$\{10,3\}*360$ (reg.)
	3	3^{1}	$\{18_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(N16.5)^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\{15,3\}*360$ (reg.)
$T_{[3,5/2]}$	1	$3^{1,2}$	$\{(5,6_p),3\}$	#3	Table 8 $K_{[5/2,3]}$ #3	N/A	$\mathrm{Tr}(\{3,5\})$
	2	$3^{1,2}$	$\{(5,20_s),3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{10,5\}_3)$
	3	3^{1}	$\{18_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{3,5\})^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{10,5\}_3)^\pi$
$T^{\zeta_2}_{[3,5/2]}$	1	$3^{1,2}$	$\{(10_s,12_s),3\}$	#3	N/A	N/A	$\mathrm{Tr}(N16.5)$
	2	$3^{1,2}$	$\{(10_s,10_s),3\}$	#4	N/A	N/A	$\{10,3\}*360$ (reg.)
	3	3^{1}	$\{18_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(N16.5)^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\{15,3\}*360$ (reg.)
$T_{[5,3]}$	1	$3^{1,2}$	$\{(3,10_p),3\}$	#3	Table 8 $K_{[3,5]}$ #1	N/A	$\mathrm{Tr}(\{5,3\})$
	2	$3^{1,2}$	$\{(3,20_s),3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{10,3\}_5)$
	3	3^{1}	$\{30_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{5,3\})^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{10,3\}_5)^\pi$
$T^{\zeta_2}_{[5,3]}$	1	$3^{1,2}$	$\{(6_s,20_s),3\}$	#3	N/A	N/A	$\mathrm{Tr}(N16.5^\delta)$
	2	$3^{1,2}$	$\{(6_s,10_s),3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{5,6\}*120c)$
	3	3^{1}	$\{30_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(N16.5^\delta)^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{5,6\}*120c)^\pi$
$T_{[5/2,3]}$	1	$3^{1,2}$	$\{(3,10_p),3\}$	#3	Table 8 $K_{[3,5/2]}$ #1	N/A	$\mathrm{Tr}(\{5,3\})$
	2	$3^{1,2}$	$\{(3,20_s),3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{10,3\}_5)$
	3	3^{1}	$\{30_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{5,3\})^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{10,3\}_5)^\pi$
$T^{\zeta_2}_{[5/2,3]}$	1	$3^{1,2}$	$\{(6_s,20_s),3\}$	#3	N/A	N/A	$\mathrm{Tr}(N16.5^\delta)$
	2	$3^{1,2}$	$\{(6_s,10_s),3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{5,6\}*120c)$
	3	3^{1}	$\{30_{rh},3\}$	#1	N/A	N/A	$\mathrm{Tr}(N16.5^\delta)^\pi$
	4	3^{1}	$\{15_r,3\}$	#2	N/A	N/A	$\mathrm{Tr}(\{5,6\}*120c)^\pi$

Table 9: 32 of the vertex-transitive 3-orbit polyhedra with symmetry group $[3,5]$ and 60 vertices

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
$T_{[5,5/2]}$	1	$3^{1,2}$	$\{(5, 10_p), 3\}$	#3	Table 8 $Cl_{[3,5/2]}$ #1	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\})$
	2	$3^{1,2}$	$\{(5, 12_s), 3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\}^\pi)$
	3	3^1	$\{30_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\})^\pi$
	4	3^1	$\{9_r, 3\}$	#2	Table 11 MD #5	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\}^\pi)^\pi$
$T^{\zeta_2}_{[5,5/2]}$	1	$3^{1,2}$	$\{(10_s, 20_s), 3\}$	#3	N/A	N/A	$\mathrm{Tr}(\{10, 10 \mid 3\}_3)$
	2	$3^{1,2}$	$\{(10_s, 6_s), 3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{3, 10\} * 120b)$
	3	3^1	$\{30_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{10, 10 \mid 3\}_3)^\pi$
	4	3^1	$\{9_r, 3\}$	#2	Table 11 MD^{ζ_2} #5	N/A	$\mathrm{Tr}(\{3, 10\} * 120b)^\pi$
$T_{[5/2,5]}$	1	$3^{1,2}$	$\{(5, 10_p), 3\}$	#3	Table 8 $Cl_{[3,5]}$ #1	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\})$
	2	$3^{1,2}$	$\{(5, 12_s), 3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\}^\pi)$
	3	3^1	$\{30_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\})^\pi$
	4	3^1	$\{9_r, 3\}$	#2	Table 11 MD^{ζ_1} #8	N/A	$\mathrm{Tr}(\{5, 5 \mid 3\}^\pi)^\pi$
$T^{\zeta_2}_{[5/2,5]}$	1	$3^{1,2}$	$\{(10_s, 20_s), 3\}$	#3	N/A	N/A	$\mathrm{Tr}(\{10, 10 \mid 3\}_3)$
	2	$3^{1,2}$	$\{(10_s, 6_s), 3\}$	#4	N/A	N/A	$\mathrm{Tr}(\{3, 10\} * 120b)$
	3	3^1	$\{30_{rh}, 3\}$	#1	N/A	N/A	$\mathrm{Tr}(\{10, 10 \mid 3\}_3)^\pi$
	4	3^1	$\{9_r, 3\}$	#2	Table 11 MD^{ζ} #8	N/A	$\mathrm{Tr}(\{3, 10\} * 120b)^\pi$

Table 10: The remaining 16 vertex-transitive 3-orbit polyhedra with symmetry group $[3, 5]$ and 60 vertices

1-skeleton	#	Class	Schläfli symbol	Petrial	Dual	Figure	Type
MD	1	$3^{1,2}$	$\{(5,12_s),9_r\}$	#5	N/A	43ae, 46b, 47b	MD^{ζ_1} #4
	2	$3^{1,2}$	$\{(3,6_p),9_r\}$	N/A	Table 8 $Cl_{[5/2,3]}$ #1	43bd, 46a, 47a	MD^{ζ_1} #3
	3	$3^{1,2}$	$\{(3,12_s),9_r\}$	N/A	N/A	43be, 46a, 47b	MD^{ζ_1} #2
	4	$3^{1,2}$	$\{(5,6_p),9_r\}$	#8	Table 8 $M_{[5/2,3]}$ #2	43cd, 46c, 47a	MD^{ζ_1} #1
	5	3^1	$\{3,9_r\}$	#1	Table 10 $T_{[5,5/2]}$ #4	44a, 48a	MD^{ζ_1} #8
	6	3^1	$\{18_{rh},9_r\}$	N/A	N/A	44b, 48b	MD^{ζ_1} #7
	7	3^1	$\{9_r,9_r\}$	N/A	N/A	44c, 48c	MD^{ζ_1} #6
	8	3^1	$\{6_{rh},9_r\}$	#4	N/A	44d, 48d	MD^{ζ_1} #5
	9	3^1	$\{6_r,9_r\}$	N/A	N/A	44e, 48e	MD^{ζ_1} #9
MD^{ζ_1}	1	$3^{1,2}$	$\{(5,6_p),9_r\}$	#5	Table 8 $M_{[5,3]}$ #2	43ae	MD #4
	2	$3^{1,2}$	$\{(3,12_s),9_r\}$	N/A	N/A	43bd	MD #3
	3	$3^{1,2}$	$\{(3,6_p),9_r\}$	N/A	Table 8 $Cl_{[5,3]}$ #1	43be	MD #2
	4	$3^{1,2}$	$\{(5,12_s),9_r\}$	#8	N/A	43cd	MD #1
	5	3^1	$\{6_{rh},9_r\}$	#1	N/A	44a	MD #8
	6	3^1	$\{9_r,9_r\}$	N/A	N/A	44b	MD #7
	7	3^1	$\{18_{rh},9_r\}$	N/A	N/A	44c	MD #6
	8	3^1	$\{3,9_r\}$	#4	Table 10 $T_{[5/2,5]}$ #4	44d	MD #5
	9	3^1	$\{6_r,9_r\}$	N/A	N/A	44f, 49	MD #9
MD^{ζ_2}	1	$3^{1,2}$	$\{(10_s,6_s),9_r\}$	#5	N/A	43ae	MD^{ζ} #4
	2	$3^{1,2}$	$\{(6_s,12_s),9_r\}$	N/A	N/A	43bd	MD^{ζ} #3
	3	$3^{1,2}$	$\{(6_s,6_s),9_r\}$	N/A	N/A	43be	MD^{ζ} #2
	4	$3^{1,2}$	$\{(10_s,12_s),9_r\}$	#8	N/A	43cd	MD^{ζ} #1
	5	3^1	$\{3,9_r\}$	#1	Table 10 $T^{\zeta_2}_{[5,5/2]}$ #4	44a	MD^{ζ} #8
	6	3^1	$\{18_{rh},9_r\}$	N/A	N/A	44b	MD^{ζ} #7
	7	3^1	$\{9_r,9_r\}$	N/A	N/A	44c	MD^{ζ} #6
	8	3^1	$\{6_{rh},9_r\}$	#4	N/A	44d	MD^{ζ} #5
	9	3^1	$\{6_r,9_r\}$	N/A	N/A	44e	MD^{ζ} #9
MD^{ζ}	1	$3^{1,2}$	$\{(10_s,12_s),9_r\}$	#5	N/A	43ae	MD^{ζ_2} #4
	2	$3^{1,2}$	$\{(6_s,6_s),9_r\}$	N/A	N/A	43bd	MD^{ζ_2} #3
	3	$3^{1,2}$	$\{(6_s,12_s),9_r\}$	N/A	N/A	43be	MD^{ζ_2} #2
	4	$3^{1,2}$	$\{(10_s,6_s),9_r\}$	#8	N/A	43cd	MD^{ζ_2} #1
	5	3^1	$\{6_{rh},9_r\}$	#1	N/A	44a	MD^{ζ_2} #8
	6	3^1	$\{9_r,9_r\}$	N/A	N/A	44b	MD^{ζ_2} #7
	7	3^1	$\{18_{rh},9_r\}$	N/A	N/A	44c	MD^{ζ_2} #6
	8	3^1	$\{3,9_r\}$	#4	Table 10 $T^{\zeta_2}_{[5/2,5]}$ #4	44d	MD^{ζ_2} #5
	9	3^1	$\{6_r,9_r\}$	N/A	N/A	44f	MD^{ζ_2} #9

Table 11: The 36 vertex-transitive 3-orbit polyhedra with symmetry group $[3,5]$ and 20 vertices

14 Conflicts of interests

This work has no conflict of interests.

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